In[•]:=
```
Get@FileNameJoin@{NotebookDirectory[], "Calibration.m"};
```

Stephanie's bosonic particle spectra in de Sitter: PSALTer spectral analysis

Connection to Part III Project: In the previous script, we explored the scalar and GR theories linearised in the vicinity of the de Sitter background. The scalar theory was fairly easy to understand, but GR looked like more of a hot mess than the Fierz-Pauli linearisation. In this script, we'll confirm using PSALTer what the spectra of both theories actually are. Be advised that PSALTer is not (yet) built for the de Sitter analysis, so the evaluation won't run to completion.

First we load PSALTer.

```
------------------------------------------------------------

Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}

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Connecting to external linux executable...

Connection established.

------------------------------------------------------------

Package xAct`xTensor`  version 1.2.0, {2021, 10, 17}

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------------------------------------------------------------

Package xAct`xPert`  version 1.0.6, {2018, 2, 28}

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  and Guillermo A. Mena Marugan, under the General Public License.

** Variable $PrePrint assigned value ScreenDollarIndices

** Variable $CovDFormat changed from Prefix to Postfix

** Option AllowUpperDerivatives of ContractMetric changed from False to True

** Option MetricOn of MakeRule changed from None to All

** Option ContractMetrics of MakeRule changed from False to True

------------------------------------------------------------

Package xAct`Invar`  version 2.0.5, {2013, 7, 1}
```

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D. Yllanes and R. Portugal, under the General Public License.

** DefConstantSymbol: Defining constant symbol sigma.

** DefConstantSymbol: Defining constant symbol dim.

** Option CurvatureRelations of DefCovD changed from True to False

** Variable $CommuteCovDsOnScalars changed from True to False

----

Package xAct`xCoba` version 0.8.6, {2021, 2, 28}

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Jose M. Martin-Garcia, under the General Public License.

Package xAct`SymManipulator` version 0.9.5, {2021, 9, 14}

CopyRight (C) 2011-2021, Thomas Bäckdahl, under the General Public License.

Package xAct`xTras` version 1.4.2, {2014, 10, 30}

CopyRight (C) 2012-2014, Teake Nutma, under the General Public License.

** Variable $CovDFormat changed from Postfix to Prefix

** Option CurvatureRelations of DefCovD changed from False to True

----

Package xAct`PSALTer` version 1.0.0-developer, {2023, 5, 3}

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Linearised Klein-Gordon theory

Remember in the last script how our Weyl rescaling initially gave us an odd-looking Lagrangian in Eq. (2), and a conventional (but pathological-looking) Lagrangian in Eq. (4)? We know what PSALTer will have to say about Eq. (4), but it will be satisfying if we can draw the same conclusions by feeding in Eq. (2). We write this as follows.

$$\alpha_{1} \left(\mathcal{H}^2 \varphi^2 - 2\mathcal{H}\varphi n_{a} \partial^{a}\varphi + \partial_{a}\varphi \partial^{a}\varphi\right) \quad (1)$$

Now we try computing the particle spectrum of Eq. (1) by feeding it into PSALTer.

The (possibly singular) a-matrices associated with the Lagrangian, as defined below Eq. (18) of arXiv:1812.02675:

$$\left\{\left(\alpha_{\dot{1}}\left(k^2+2\mathcal{H}^2\right)\right),\ (0)\right\}$$

Gauge constraints on source currents:

{}

The Drazin (Moore-Penrose) inverses of these a-matrices, which are functionally analogous to the inverse b-matrices described below Eq. (21) of arXiv:1812.02675:

$$\left\{\left(\frac{1}{\alpha_{\dot{1}}\, k^2+2\,\alpha_{\dot{1}}\,\mathcal{H}^2}\right),\ (0)\right\}$$

Square masses:

$$\left\{\left\{-2\mathcal{H}^2\right\},\ \{\}\right\}$$

Massive pole residues:

$$\left\{\left\{\frac{1}{\alpha_{\dot{1}}}\right\},\ \{\}\right\}$$

Massless eigenvalues:

{}

Overall unitarity conditions:

False

Connection to Part III Project: Okay, so this is quite nice! We can see that PSALTer recognises Eq. (1) for what it is claiming to be (i.e. a tachyon). It is important that you don't write in your report that there is actually a tachyon here, only that the naive spectral method identifies one, and that this tachyon should be correctly interpreted as a healthy de Sitter mode function. The unitarity condition is false, because of course the `mass' would strictly be imaginary for real comoving Hubble scale.

Linearised general relativity

Rules {1} have been declared as UpValues for V.

Understanding the gauge constraints

Recall that in Eq. (28) and Eq. (29) of the previous script we obtained what we thought were the gauge constraints on the source stress-energy tensor: we will now import these and decompose them using the PSALTer spin-parity modes. First we consider the timelike part of the full four-vector constraint in Eq. (28).

$$\mathcal{H}\,{}^{0^+}_{\cdot}\mathcal{T}^{\parallel}+i\,k\,{}^{0^+}_{\cdot}\mathcal{T}^{\perp} \tag{2}$$

Okay, so Eq. (2) just relates the two scalar parts of $\mathcal{T}_{mn}$, and we see that if we take the Minkowski limit of vanishing $\mathcal{H}$ then we get the usual scalar source constraint of Fierz-Pauli. Next we'll target Eq. (29) from the previous script, i.e. the scalar part of the source constraints which was covariantly obtained by taking an extra derivative.

$$-i\, k\, \mathcal{H}\, {}^{0^+}_{\cdot}\mathcal{T}^{\parallel} + \mathcal{H}^2\, {}^{0^+}_{\cdot}\mathcal{T}^{\parallel} + k^2\, {}^{0^+}_{\cdot}\mathcal{T}^{\perp} \tag{3}$$

So in Eq. (3) we see exactly what we expect from Eq. (2) if we take another derivative in momentum space and remember how the Hubble scale increases its power by one if it is differentiated. This is a nice cross-check of Eq. (2), since it was obtained using a slightly different method.

Connection to Part III Project: The important observation here is that the scalar part of the source constraints survives on de Sitter, and it picks up an extra term so as to relate the two scalar parts of the stress-energy tensor. It is also true that the parity-odd vector part of the Fierz-Pauli source constraint survives in full, without any modifications (though we won't bother to show it here).

Performing the spectral analysis

Now we move on to consider the spectral analysis of the theory in Eq. (33) of the previous script, i.e. the hard-looking linearisation of GR on the de Sitter background. Again we import the string, and in PSALTer terms here is the resulting theory.

$$\begin{aligned} &\underset{1}{\alpha.}\mathcal{H}^2\, h_{ab}\, h^{ab} - \underset{1}{\alpha.}\mathcal{H}^2\, h^{a}{}_{a}\, h^{b}{}_{b} - 2\,\underset{1}{\alpha.}\mathcal{H}^2\, h_{a}{}^{c}\, h_{bc}\, n^{a}\, n^{b} - \underset{1}{\alpha.}\mathcal{H}^2\, h_{ab}\, h^{c}{}_{c}\, n^{a}\, n^{b} + \\ &\underset{1}{\alpha.}\mathcal{H}\, h_{a}{}^{b}\, n^{a}\, \partial_b h^{c}{}_{c} - \frac{1}{2}\,\underset{1}{\alpha.}\, h^{ab}\, \partial_b\partial_a h^{c}{}_{c} + \underset{1}{\alpha.}\mathcal{H}\, h^{bc}\, n^{a}\, \partial_c h_{ab} - \underset{1}{\alpha.}\mathcal{H}\, h^{b}{}_{b}\, n^{a}\, \partial_c h_{a}{}^{c} - \\ &\underset{1}{\alpha.}\mathcal{H}\, h_{a}{}^{b}\, n^{a}\, \partial_c h_{b}{}^{c} + \underset{1}{\alpha.}\, h^{ab}\, \partial_c\partial_b h_{a}{}^{c} - \frac{1}{2}\,\underset{1}{\alpha.}\, h^{a}{}_{a}\, \partial_c\partial_b h^{bc} - \frac{1}{2}\,\underset{1}{\alpha.}\, h^{ab}\, \partial_c\partial^{c} h_{ab} + \frac{1}{2}\,\underset{1}{\alpha.}\, h^{a}{}_{a}\, \partial_c\partial^{c} h^{b}{}_{b} \end{aligned} \tag{4}$$

Now we feed Eq. (4) into the PSALTer package.

The (possibly singular) a-matrices associated with
the Lagrangian, as defined below Eq. (18) of arXiv:1812.02675:

$$\left\{\begin{pmatrix} -3\,\alpha_{\dot{1}}\,\mathcal{H}^2 & \frac{1}{2}\sqrt{3}\,\alpha_{\dot{1}}\left(2\,i\,k-3\,\mathcal{H}\right)\mathcal{H} \\ \frac{1}{2}\sqrt{3}\,\alpha_{\dot{1}}\left(-2\,i\,k-3\,\mathcal{H}\right)\mathcal{H} & -\alpha_{\dot{1}}\left(k^2+2\,\mathcal{H}^2\right) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0), (0), \left(\frac{1}{2}\,\alpha_{\dot{1}}\left(k^2+2\,\mathcal{H}^2\right)\right), (0)\right\}$$

Gauge constraints on source currents:

$$\left\{{}^{1^-}_{\cdot}\mathcal{T}^{\perp a} == 0\right\}$$

The Drazin (Moore-Penrose) inverses of these a-matrices, which are functionally
analogous to the inverse b-matrices described below Eq. (21) of arXiv:1812.02675:

$$\left\{\begin{pmatrix} \frac{4\left(k^2+2\,\mathcal{H}^2\right)}{3\,\alpha_{\dot{1}}\,\mathcal{H}^4} & \frac{4\,i\,k-6\,\mathcal{H}}{\sqrt{3}\,\alpha_{\dot{1}}\,\mathcal{H}^3} \\ \frac{-4\,i\,k-6\,\mathcal{H}}{\sqrt{3}\,\alpha_{\dot{1}}\,\mathcal{H}^3} & \frac{4}{\alpha_{\dot{1}}\,\mathcal{H}^2} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0), (0), \left(\frac{2}{\alpha_{\dot{1}}\,k^2+2\,\alpha_{\dot{1}}\,\mathcal{H}^2}\right), (0)\right\}$$

Square masses:

$$\left\{\{\}, \{\}, \{\}, \{\}, \left\{-2\,\mathcal{H}^2\right\}, \{\}\right\}$$

Massive pole residues:

$$\left\{\{\}, \{\}, \{\}, \{\}, \left\{\frac{2}{\alpha_{\dot{1}}}\right\}, \{\}\right\}$$

Massless eigenvalues:

Eigenvalues[(Symmetric → False)⟦1 ;; 1, 2 ;; 2⟧]

Overall unitarity conditions:

False

Connection to Part III Project: Okay, so this is very interesting. The first observation to make is that the square mass is precisely the same as that which we obtained for the scalar theory, i.e. it suggests the same tachyonic character even down to numerical factors. This is satisfying, and pretty important, because there is a `lore' that in de Sitter the graviton polarisations both inherit precisely the same mode functions as the free scalar. The remaining task is to show that only two such polarisations are actually propagating. This is quite tricky to achieve using the existing PSALTer setup for the following reason. If we accept that the Hubble scale is acting somewhat like a (tachyonic) mass, then we notice in the above spectrum that only the spin-two parity-even tensor part of the theory is propagating at all (look at the denominator: only that sector has any momentum underneath). Usually we would then say that five d.o.f are moving in the tensor sector (remember how the Wigner little group affects d.o.f for a spin state when you go from massless to massive), so what are we missing? The point is that in the massless case we remember how we need to go over into an explicit coordinate basis and incorporate the source constraints in order to find the massless eigenvalues in the limit of the null cone: I believe we would now have to do the same in the limit of the tachyonic one-sheet hyperbola. Frankly, PSALTer isn't set up to take that new limit (it would require a few weeks' work), but we have a shortcut as follows. It is enough to notice that from Eqs. (2), and (3), along with the (unwritten) vector equation, that the four diffeomorphism gauge symmetries of GR survive and flourish in the linear model on de Sitter. Consequently, we know that the whole spectrum above cannot have more than two eigenvalues over the new (apparently tachyonic) pole, because ten take two times four equals two (Google the famous phrase `the gauge always hits twice'). With this observation, the analysis of GR on the de Sitter background is concluded.

Connection to Part III Project: One final point to make, in the context of the apparent mass interpretation of the mode functions, is regards Fierz-Pauli tuning. As I mentioned in my calibration script some months ago, one has to be extremely careful in adding mass terms to the Fierz-Pauli model so that only the parity-even spin-two mode becomes massive. Most mass terms one adds covariantly also activate a mode from the scalar sector (Google Boulware-Deser ghost). There is one acceptable mass term, which is called the (tuned) Fierz-Pauli mass term, and which can avoid introducing the a new parity-even spin-zero pole. Astonishingly, the structure of the spin-zero sector in the above de Sitter model bears a striking resemblance to the tuned massive gravity spectrum (please go and compare with my previous calibration script). I'm not sure if this has ever been noticed by other authors, but is certainly worth a comment in your report. It is as if the de Sitter background is playing the role of Fierz-Pauli tuning, so that the mode function pseudo-tachyon does not crop up in the scalar sector.