In[ ]:=
```
Get@FileNameJoin@{NotebookDirectory[], "CurvedFierzPauli.m"};
```

Stephanie's bosonic particle spectra in de Sitter: theoretical development

Connection to Part III Project: The aim here is to explore how the standard particle spectrum/saturated propagator algorithm can be applied in a de Sitter spacetime. We will explore the case of a massless scalar field, and GR linearised in the presence of a cosmological constant.

```
------------------------------------------------------------

Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}

CopyRight (C) 2003-2020, Jose M. Martin-Garcia, under the General Public License.

Connecting to external linux executable...

Connection established.

------------------------------------------------------------

Package xAct`xTensor`  version 1.2.0, {2021, 10, 17}

CopyRight (C) 2002-2021, Jose M. Martin-Garcia, under the General Public License.

------------------------------------------------------------

These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.

------------------------------------------------------------

------------------------------------------------------------

Package xAct`xPert`  version 1.0.6, {2018, 2, 28}

CopyRight (C) 2005-2020, David Brizuela, Jose M. Martin-Garcia
  and Guillermo A. Mena Marugan, under the General Public License.

------------------------------------------------------------

These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.

------------------------------------------------------------

** Variable $PrePrint assigned value ScreenDollarIndices
```

```
** Variable $CovDFormat changed from Prefix to Postfix

** Option AllowUpperDerivatives of ContractMetric changed from False to True

** Option MetricOn of MakeRule changed from None to All

** Option ContractMetrics of MakeRule changed from False to True

------------------------------------------------------------

Package xAct`Invar`  version 2.0.5, {2013, 7, 1}

CopyRight (C) 2006–2020, J. M. Martin-Garcia,
  D. Yllanes and R. Portugal, under the General Public License.

** DefConstantSymbol: Defining constant symbol sigma.

** DefConstantSymbol: Defining constant symbol dim.

** Option CurvatureRelations of DefCovD changed from True to False

** Variable $CommuteCovDsOnScalars changed from True to False

------------------------------------------------------------

Package xAct`xCoba`  version 0.8.6, {2021, 2, 28}

CopyRight (C) 2005–2021, David Yllanes and
  Jose M. Martin-Garcia, under the General Public License.

------------------------------------------------------------

Package xAct`SymManipulator`  version 0.9.5, {2021, 9, 14}

CopyRight (C) 2011–2021, Thomas Bäckdahl, under the General Public License.

------------------------------------------------------------

Package xAct`xTras`  version 1.4.2, {2014, 10, 30}

CopyRight (C) 2012–2014, Teake Nutma, under the General Public License.

** Variable $CovDFormat changed from Postfix to Prefix

** Option CurvatureRelations of DefCovD changed from False to True

------------------------------------------------------------

These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.

------------------------------------------------------------
```

Define a manifold and curved metric.

```
** DefManifold: Defining manifold M4.

** DefVBundle: Defining vbundle TangentM4.

** DefTensor: Defining symmetric metric tensor G[-a, -c].

** DefTensor: Defining antisymmetric tensor epsilonG[-a, -b, -c, -d].
```

```
** DefTensor: Defining tetrametric TetraG[-a, -b, -c, -d].
** DefTensor: Defining tetrametric TetraG†[-a, -b, -c, -d].
** DefCovD: Defining covariant derivative CD[-a].
** DefTensor: Defining vanishing torsion tensor TorsionCD[a, -b, -c].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[a, -b, -c].
** DefTensor: Defining Riemann tensor RiemannCD[-a, -b, -c, -d].
** DefTensor: Defining symmetric Ricci tensor RicciCD[-a, -b].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-a, -b].
** DefTensor: Defining Weyl tensor WeylCD[-a, -b, -c, -d].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-a, -b].
** DefTensor: Defining Kretschmann scalar KretschmannCD[].
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToTFRicci for dim 4
** DefCovD:  Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining symmetrized Riemann tensor SymRiemannCD[-a, -b, -c, -d].
** DefTensor: Defining symmetric Schouten tensor SchoutenCD[-a, -b].
** DefTensor: Defining symmetric cosmological Schouten tensor SchoutenCCCD[LI[_], -a, -b].
** DefTensor: Defining symmetric cosmological Einstein tensor EinsteinCCCD[LI[_], -a, -b].
** DefTensor: Defining weight +2 density DetG[]. Determinant.
** DefParameter: Defining parameter PerturbationParameterG.
** DefTensor: Defining tensor PerturbationG[LI[order], -a, -b].
** DefConstantSymbol: Defining constant symbol EinsteinConstant.
** DefConstantSymbol: Defining constant symbol CosmologicalConstant.
** DefTensor: Defining tensor ScaleFactor[].
** DefConstantSymbol: Defining constant symbol HubbleNumber.
** DefConstantSymbol: Defining constant symbol HubbleScale.
** DefTensor: Defining tensor V[-a].
   Rules {1, 2} have been declared as UpValues for V.
   Rules {1} have been declared as UpValues for V.
```

```
Rules {1} have been declared as UpValues for ScaleFactor.
```

Define the Einstein constant κ the scale factor a, Hubble number (constant) H, cosmological constant Λ, comoving Hubble number $\mathcal{H}$, and the n_m-vector which will be unit-timelike in the flat spacetime to which de Sitter is conformal. Note that the unit-timelike vector will have downstairs valence, so that it naturally inherits the tensor character of derivative operators later on without the need for a metric.

Linearised Klein-Gordon theory

```
** DefTensor: Defining tensor Phi[].
```

In the first instance we consider a massless, shift-symmetric Klein-Gordon field. We allow the background spacetime to become de Sitter.

$$\frac{1}{2}\sqrt{-\tilde{\tilde{g}}}\,\nabla_a\phi\nabla^a\phi \tag{1}$$

By a Weyl-rescaling purely of the scalar field (but not of the metric), we can of course obtain from Eq. (1) the following version of the theory.

$$\frac{1}{2}H^2\phi^2 a^2 - H\phi a\, n^a\,\nabla_a\phi + \frac{1}{2}\nabla_a\phi\nabla^a\phi \tag{2}$$

Note that in Eq. (2) the covariant derivative is really the partial derivative on the (fictional) flat spacetime, whose coordinates are the conformal coordinates of the de Sitter spacetime. The form of the theory looks a bit cryptic still, but by an additional surface term we can obtain from Eq. (2) the following version.

$$H^2\phi^2 a^2 + \frac{1}{2}\nabla_a\phi\nabla^a\phi \tag{3}$$

Okay, so Eq. (3) is now looking much better. Let's finally move to the comoving Hubble.

$$\mathcal{H}^2\phi^2 + \frac{1}{2}\nabla_a\phi\nabla^a\phi \tag{4}$$

Connection to Part III Project: So we see that Eq. (1) on de Sitter actually looks like Eq. (4) on flat space. This would seem superficially to be a disaster: something that looks like a tachyonic mass term has appeared! Actually, we know that this is okay. The mass term is time-dependent because the Hubble scale here is comoving; as a result the evolution of the system can be found using the standard Bessel method. This is all consistent with what we know about the cosmological mode functions. However the tachyonic mass interpretation will be really important for putting together a consistent de Sitter particle spectrum dictionary for more general models. Next, we'll look at the graviton.

Linearised general relativity

```
** DefTensor: Defining tensor LinearMetric[-a, -b].
** DefTensor: Defining tensor LinearMetric†[-a, -b].
```

Define a tensor h_{ab} which acts as the linear metric.

```
** DefTensor: Defining tensor TraceReversedLinearMetric[-a, -b].
** DefTensor: Defining tensor TraceReversedLinearMetric†[-a, -b].
** DefTensor: Defining tensor TraceLinearMetric[].
** DefTensor: Defining tensor TraceLinearMetric†[].
```

Define the trace-reversed metric perturbation $\bar{h}_{ab}$ and the trace of the metric perturbation h.

```
Rules {1, 2} have been declared as DownValues for TraceReversedLinearMetric.
```

Now we define the Einstein-Hilbert action. We're going to have to be quite careful, and build the quadratic aspect of the Lagrangian density up slowly using variational derivatives.

Background terms

First term, the Hilbert term.

$$-\frac{\sqrt{-\tilde{\tilde{g}}}\, R[\nabla]}{2\,\kappa} \tag{5}$$

Second term, the source.

$$-\frac{\Lambda \sqrt{-\tilde{\tilde{g}}}}{\kappa} \tag{6}$$

First order terms

The first order correction to Eq. (5) is as follows.

$$\frac{\sqrt{-\tilde{\tilde{g}}}\left(G[\nabla]^{a}{}_{a}\, h + 2\, G[\nabla]^{ab}\, \bar{h}_{ab}\right)}{4\,\kappa} \tag{7}$$

The first order correction to Eq. (6) is as follows.

$$-\frac{\Lambda \sqrt{-\tilde{\tilde{g}}}\, h}{2\,\kappa} \tag{8}$$

Second order terms

The second order correction to Eq. (5) is as follows.

$$\frac{\sqrt{-\tilde{\bar{g}}}\left(R[\nabla]\,h^2 - 2\,R[\nabla]\;\bar{h}_{ab}\;\bar{h}^{ab} - 4\;G[\nabla]^{ab}\left(h\;\bar{h}_{ab} + 2\;\bar{h}_a{}^c\;\bar{h}_{bc}\right) + h\,\nabla_a\nabla^a h + 4\;\bar{h}^{ab}\;\nabla_c\nabla_b\bar{h}_a{}^c - 2\;\bar{h}^{ab}\;\nabla_c\nabla^c\bar{h}_{ab}\right)}{8\,\kappa} \tag{9}$$

The second order correction to Eq. (6) is as follows.

$$-\frac{\Lambda\,\sqrt{-\tilde{\bar{g}}}\left(h^2 - 2\;\bar{h}_{ab}\;\bar{h}^{ab}\right)}{4\,\kappa} \tag{10}$$

```
** DefTensor: Defining tensor Gen[a].
```

Now define a generating vector $\mathcal{V}^a$ for diffeomorphisms of the background. The Lie derivative induces the following change to h_{ab} at first order.

$$\nabla_a\mathcal{V}_b + \nabla_b\mathcal{V}_a \tag{11}$$

And the following change at second order.

$$h^c{}_b\;\nabla_a\mathcal{V}_c + h^c{}_a\;\nabla_b\mathcal{V}_c + \mathcal{V}^c\;\nabla_c h_{ab} \tag{12}$$

The next step is to transform Eqs. (7), (8), (9), and (10) using the first and second order changes Eqs. (11), and (12), to lowest order in $\mathcal{V}^a$, and then take the variational derivative of the transformed quantity (integrated over the spacetime) with respect to $\mathcal{V}^a$.

First order terms

Variation of Eq. (7) using Eq. (11).

$$0 \tag{13}$$

Variation of Eq. (7) using Eq. (12).

$$\frac{\sqrt{-\tilde{\bar{g}}}\;\mathcal{V}^a\left(G[\nabla]^b{}_b\;\nabla_a h - 2\;G[\nabla]_{ab}\;\nabla^b h + 2\;G[\nabla]^{bc}\left(\nabla_a\bar{h}_{bc} - 2\,\nabla_c\bar{h}_{ab}\right)\right)}{4\,\kappa} \tag{14}$$

Variation of Eq. (8) using Eq. (11).

$$0 \tag{15}$$

```
TensorTheory
```

Variation of Eq. (8) using Eq. (12).

$$\frac{\Lambda \sqrt{-\tilde{\bar{g}}}\ \mathcal{V}^{a}\ \nabla_{b}\bar{h}_{a}{}^{b}}{\kappa} \tag{16}$$

Second order terms

Variation of Eq. (9) using Eq. (11).

$$-\frac{\sqrt{-\tilde{\bar{g}}}\ \mathcal{V}^{a}\ \left(G[\nabla]^{b}{}_{b}\ \nabla_{a}h - 2\ G[\nabla]_{ab}\ \nabla^{b}h + 2\ G[\nabla]^{bc}\ \left(\nabla_{a}\bar{h}_{bc} - 2\nabla_{c}\bar{h}_{ab}\right)\right)}{2\,\kappa} \tag{17}$$

Variation of Eq. (10) using Eq. (11).

$$-\frac{2\Lambda \sqrt{-\tilde{\bar{g}}}\ \mathcal{V}^{a}\ \nabla_{b}\bar{h}_{a}{}^{b}}{\kappa} \tag{18}$$

So we see in Eq. (13) and by comparing Eq. (14) with Eq. (17), and also in Eq. (15) and by comparing Eq. (16) with Eq. (18), that there is cancellation across orders (don't forget the factor of two from the perturbative expansion) and within the gravitational and source sectors individually due to Bianchi and the conservation of stress-energy (which are separately true).

Now we take Eq. (17) and we confine to the background shell.

$$-\frac{2\Lambda \sqrt{-\tilde{\bar{g}}}\ \mathcal{V}^{a}\ \nabla_{b}\bar{h}_{a}{}^{b}}{\kappa} \tag{19}$$

So from Eq. (19) we notice that Eq. (14) and Eq. (16), as well as Eq. (17) and Eq. (18), would cancel across the matter and gravity sectors but within each order, just due to the background shell.

Weyl transformation

Let's check that the Weyl-transformed first-order operator is satisfying the background field equations. We add Eq. (7) to Eq. (8) to obtain the first order part of the whole free-space Lagrangian, and then we rescale the metric perturbation h_{mn} according to our rules.

$$-\frac{\Lambda \sqrt{-\tilde{\bar{g}}}\ h_{a}{}^{a}}{2\,\kappa} + \frac{\sqrt{-\tilde{\bar{g}}}\ \left(2\ h_{ab}\ \ R[\nabla]^{ab} - h_{a}{}^{a}\ R[\nabla]\right)}{4\,\kappa} \tag{20}$$

$$-\frac{\left(\Lambda - 3\,H^{2}\right)\ h^{a}{}_{a}\ a^{3}}{2\,\kappa} == 0 \tag{21}$$

So, in Eq. (21) this is what we expect. We ought to get $\Lambda == 3\ \mathrm{H}^{2}$ on the background shell, so the first order part of the Lagrangian vanishes when the background field equations are imposed, and more-over this remains true even once we have rescaled the h_{mn} field.

Now we repeat the above process but at second order. Here is what we believe to be the total wave operator (i.e. obtained by adding Eq. (9) with Eq. (10)) when expressed in terms of the perturbed field and the curvature (i.e. not trace-reverse or Einstein tensor).

$$\frac{\Lambda\sqrt{-\tilde{\tilde{g}}}\left(2 h_{ab} h^{ab} - h_a{}^a h^b{}_b\right)}{4\kappa} + \frac{\sqrt{-\tilde{\tilde{g}}}\, h_{ab} h^c{}_c R[\nabla]^{ab}}{4\kappa} + \frac{\sqrt{-\tilde{\tilde{g}}}\, h_a{}^a h^{bc} R[\nabla]_{bc}}{4\kappa} - \frac{\sqrt{-\tilde{\tilde{g}}}\, h_{ab} h^a{}_c R[\nabla]^{bc}}{\kappa} + \frac{\sqrt{-\tilde{\tilde{g}}}\, h_{ab} h^{ab} R[\nabla]}{4\kappa} - \frac{\sqrt{-\tilde{\tilde{g}}}\, h_a{}^a h^b{}_b R[\nabla]}{8\kappa} - \frac{\sqrt{-\tilde{\tilde{g}}}\, h_a{}^b \nabla_b\nabla^a h^c{}_c}{8\kappa} + \frac{\sqrt{-\tilde{\tilde{g}}}\, h_a{}^a \nabla_b\nabla^b h^c{}_c}{8\kappa} - \frac{\sqrt{-\tilde{\tilde{g}}}\, h_{ab} \nabla^b\nabla^a h^c{}_c}{8\kappa} - \frac{\sqrt{-\tilde{\tilde{g}}}\, h_{ab} \nabla^b\nabla_c h^{ac}}{4\kappa} + \frac{\sqrt{-\tilde{\tilde{g}}}\, h_a{}^c \nabla_c\nabla_b h^{ab}}{4\kappa} - \frac{\sqrt{-\tilde{\tilde{g}}}\, h_a{}^a \nabla_c\nabla_b h^{bc}}{4\kappa} + \frac{\sqrt{-\tilde{\tilde{g}}}\, h_{ab} \nabla_c\nabla^b h^{ac}}{2\kappa} - \frac{\sqrt{-\tilde{\tilde{g}}}\, h_{ab} \nabla_c\nabla^c h^{ab}}{4\kappa} + \frac{\sqrt{-\tilde{\tilde{g}}}\, h_a{}^a \nabla_c\nabla^c h^b{}_b}{8\kappa} \tag{22}$$

Now we will try to look at the Weyl-transformed version of Eq. (22).

$$\frac{(\Lambda - 2H^2)\, h_{ab} h^{ab} a^2}{2\kappa} + \frac{(-\Lambda + H^2)\, h^a{}_a h^b{}_b a^2}{4\kappa} - \frac{H^2 h_a{}^c h_{bc} a^2 n^a n^b}{\kappa} - \frac{H^2 h_{ab} h^c{}_c a^2 n^a n^b}{2\kappa} + \frac{H h_a{}^b a\, n^a \nabla_b h^c{}_c}{2\kappa} - \frac{h^{ab} \nabla_b\nabla_a h^c{}_c}{4\kappa} + \frac{H h^{bc} a\, n^a \nabla_c h_{ab}}{2\kappa} - \frac{H h^b{}_b a\, n^a \nabla_c h_a{}^c}{2\kappa} - \frac{H h_a{}^b a\, n^a \nabla_c h_b{}^c}{2\kappa} + \frac{h^{ab} \nabla_c\nabla_b h_a{}^c}{2\kappa} - \frac{h^a{}_a \nabla_c\nabla_b h^{bc}}{4\kappa} - \frac{h^{ab} \nabla_c\nabla^c h_{ab}}{4\kappa} + \frac{h^a{}_a \nabla_c\nabla^c h^b{}_b}{4\kappa} == 0 \tag{23}$$

We can see that the power of the scale factor in the transformation is appropriate, since we will recover the (unscaled) Fierz-Pauli operator among the second-order terms in Eq. (23).

Now we look at the part of the variation that we keep in the second-order operator, recalling that this will now be the variation of the rescaled perturbation (hence we need to divide by the scale factor).

The rescaled transformation in Eq. (11), which transforms the rescaled h_{ab} is as follows.

$$2 H g_{ab} \mathcal{V}^c a^2 n_c + a \nabla_a \mathcal{V}_b + a \nabla_b \mathcal{V}_a \tag{24}$$

Now we wish to impose this variation in Eq. (24) on the wave operator Eq. (23).

$$\frac{H(\Lambda - 3H^2)\,\mathcal{V}^a\;h^b{}_b\;a^4\;n_a}{\kappa} - \frac{6H(\Lambda - 3H^2)\,\mathcal{V}^a\;h_{ab}\;a^4\;n^b}{\kappa} + \frac{(\Lambda - 3H^2)\,\mathcal{V}^a\;a^3\nabla_a h^b{}_b}{\kappa} - \frac{2(\Lambda - 3H^2)\,\mathcal{V}^a\;a^3\nabla_b h_a{}^b}{\kappa}$$

So once again, as with the cancellation of Eq. (14) with Eq. (16), the gauge transformation we are proposing is indeed a gauge transformation of the linear theory when we impose the background shell condition $\Lambda == 3\,\mathrm{H}^{`}$.

```
** DefTensor: Defining tensor StressEnergy[a, b].
```

We define the (purely perturbative) stress-energy tensor $\mathcal{T}^{mn}$ which is supposed to be the conjugate source to h^{mn}. Thus the second order Lagrangian Eq. (22) is augmented by the following source coupling term.

$$h_{ab}\;\mathcal{T}^{ab} \tag{26}$$

It is important to understand that Eq. (26) represents the conjugate source coupling after the Weyl transformation has already taken place; thus we are quite loose about what $\mathcal{T}_{mn}$ really means, just so long as it is the appropriate current that would appear on the fictional flat spacetime.

Now we wish to impose the variation in Eq. (24) on the source coupling Eq. (26).

$$2H\,\mathcal{V}^a\;a^2\,\mathcal{T}^b{}_b\;\;n_a - 2H\,\mathcal{V}^a\;a^2\,\mathcal{T}_{ab}\;\;n^b - 2\,\mathcal{V}^a\;a\,\nabla_b\mathcal{T}_a{}^b == 0 \tag{27}$$

Now this condition remains true for any generating vector.

$$-\mathcal{H}\,\mathcal{T}_{ma}\;\;n^a + \mathcal{H}\,\mathcal{T}^a{}_a\;\;n_m - \nabla_a\mathcal{T}_m{}^a == 0 \tag{28}$$

Connection to Part III Project: The conservation law obeyed by the effective flat-space source is encoded in Eq. (28). It is clear that this just descends from the curved-space version of the Bianchi identity, and so there are really four independent gauge constraints at play, just like there were with the Fierz-Pauli model.

But what if we want to compute the scalar part of Eq. (28) in advance?

$$\mathcal{H}^2\,\mathcal{T}^a{}_a - \mathcal{H}^2\,\mathcal{T}_{am}\;\;n^a\;\;n^m + \mathcal{H}\;n^a\;\;\nabla_a\mathcal{T}^m{}_m - \mathcal{H}\;n^a\;\;\nabla_m\mathcal{T}_a{}^m - \nabla_m\nabla_a\mathcal{T}^{am} == 0 \tag{29}$$

So, the condition Eq. (29) is also true.

Now we take the final version Eq. (29) and we convert it to a string which can be fed into a PSALTer session later. Once PSALTer is loaded, we will be able to perform the mode decomposition and discover what the constraints on the source currents really are in terms of the spin-parity parts of the fields.

"HubbleScale^2*StressEnergy[a, –a] – HubbleScale^2*StressEnergy[–a, –m]*V[a]*V[m] + HubbleScale*V[a]*CD[–a][StressEnergy[m, –m]] – HubbleScale*V[a]*CD[–m][StressEnergy[–a, m]] – CD[–m][CD[–a][StressEnergy[a, m]]]" (30)

Now we take the other version Eq. (28) and we do precisely the same thing.

"–(HubbleScale*StressEnergy[–m, –a]*V[a]) + HubbleScale*StressEnergy[a, –a]*V[–m] – CD[–a][StressEnergy[–m, a]]" (31)

Linear theory for PSALTer

Here is the on-shell version of the Lagrangian in Eq. (23) which we will export to be used by PSALTer in the calibration script.

$$\frac{\mathcal{H}^2 h_{ab} h^{ab}}{2\kappa} - \frac{\mathcal{H}^2 h^a{}_a h^b{}_b}{2\kappa} - \frac{\mathcal{H}^2 h_a{}^c h_{bc} n^a n^b}{\kappa} - \frac{\mathcal{H}^2 h_{ab} h^c{}_c n^a n^b}{2\kappa} + \frac{\mathcal{H} h_a{}^b n^a \nabla_b h^c{}_c}{2\kappa} - \frac{h^{ab} \nabla_b \nabla_a h^c{}_c}{4\kappa} + \frac{\mathcal{H} h^{bc} n^a \nabla_c h_{ab}}{2\kappa} - \frac{\mathcal{H} h^b{}_b n^a \nabla_c h_a{}^c}{2\kappa} - \frac{\mathcal{H} h_a{}^b n^a \nabla_c h_b{}^c}{2\kappa} + \frac{h^{ab} \nabla_c \nabla_b h_a{}^c}{2\kappa} - \frac{h^a{}_a \nabla_c \nabla_b h^{bc}}{4\kappa} - \frac{h^{ab} \nabla_c \nabla^c h_{ab}}{4\kappa} + \frac{h^a{}_a \nabla_c \nabla^c h^b{}_b}{4\kappa} \tag{32}$$

Here is the alternative `wave operator' version of Eq. (32). To obtain it, we take the variational derivative of Eq. (32) with respect to h_{mn} and then post-multiply with h_{mn} to form a new scalar.

$$\frac{\mathcal{H}^2 h_{ab} h^{ab}}{\kappa} - \frac{\mathcal{H}^2 h^a{}_a h^b{}_b}{\kappa} - \frac{2\mathcal{H}^2 h_a{}^c h_{bc} n^a n^b}{\kappa} - \frac{\mathcal{H}^2 h_{ab} h^c{}_c n^a n^b}{\kappa} + \frac{\mathcal{H} h_a{}^b n^a \nabla_b h^c{}_c}{\kappa} - \frac{h^{ab} \nabla_b \nabla_a h^c{}_c}{2\kappa} + \frac{\mathcal{H} h^{bc} n^a \nabla_c h_{ab}}{\kappa} - \frac{\mathcal{H} h^b{}_b n^a \nabla_c h_a{}^c}{\kappa} - \frac{\mathcal{H} h_a{}^b n^a \nabla_c h_b{}^c}{\kappa} + \frac{h^{ab} \nabla_c \nabla_b h_a{}^c}{\kappa} - \frac{h^a{}_a \nabla_c \nabla_b h^{bc}}{2\kappa} - \frac{h^{ab} \nabla_c \nabla^c h_{ab}}{2\kappa} + \frac{h^a{}_a \nabla_c \nabla^c h^b{}_b}{2\kappa} \tag{33}$$

Now we take the final version Eq. (33) and we convert it to a string which can be fed into PSALTer.

"(HubbleScale^2*LinearMetric[–a, –b]*LinearMetric[a, b])/EinsteinConstant – (HubbleScale^2*LinearMetric[a, –a]*LinearMetric[b, –b])/EinsteinConstant – (2*HubbleScale^2*LinearMetric[–a, c]*LinearMetric[–b, –c]*V[a]*V[b])/EinsteinConstant – (HubbleScale^2*LinearMetric[–a, –b]*LinearMetric[c, –c]*V[a]*V[b])/EinsteinConstant

```
+ (HubbleScale*LinearMetric[-a, b]*V[a]*CD[-b][LinearMetric[c, -c]])/EinsteinConstant
- (LinearMetric[a, b]*CD[-b][CD[-a][LinearMetric[c, -c]]])/(2*EinsteinConstant) +
(HubbleScale*LinearMetric[b, c]*V[a]*CD[-c][LinearMetric[-a, -b]])/EinsteinConstant -
(HubbleScale*LinearMetric[b, -b]*V[a]*CD[-c][LinearMetric[-a, c]])/EinsteinConstant -
(HubbleScale*LinearMetric[-a, b]*V[a]*CD[-c][LinearMetric[-b, c]])/EinsteinConstant
+ (LinearMetric[a, b]*CD[-c][CD[-b][LinearMetric[-a, c]]])/EinsteinConstant -
(LinearMetric[a, -a]*CD[-c][CD[-b][LinearMetric[b, c]]])/(2*EinsteinConstant) -
(LinearMetric[a, b]*CD[-c][CD[c][LinearMetric[-a, -b]]])/(2*EinsteinConstant) +
(LinearMetric[a, -a]*CD[-c][CD[c][LinearMetric[b, -b]]])/(2*EinsteinConstant)"
```

Connection to Part III Project: In Eqs. (34), (31), and (30) we have some expressions that we'll study later in a PSALTer session. The key observation in this script is that the free scalar theory on de Sitter spacetime picks up what looks like a tachyonic mass in Eq. (4). This is emphatically not a tachyon, but the fact that it looks like one allows us to smoothly make contact with the saturated propagator and spin-projection particle spectrum method. When we extend to the far more complex case of dynamical gravity on the de Sitter background, our linear Lagrangian becomes Eq. (32). Comparing back to Eq. (4), we notice that Eq. (32) is fairly obfuscated. It will be our job in the PSALTer session to show that it actually gives the same spectrum as normal GR, but with the massless pole shifted to precisely the same tachyonic mass as appears in Eq. (4).