```
In[∘]:= (**)Get[FileNameJoin[{NotebookDirectory[], "xCobaCalculations.m"}]];(**)
(*Get[FileNameJoin[{NotebookDirectory[],"MA_VarD_1_.m"}]];*)
```

We want to avoid the GUI and program entirely within vim...

...that's better. Now commentary written in vim will appear in blue, and these are the lines you should pay closer attention to.

Setting up geometric preliminaries

```
------------------------------------------------------------
Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}
CopyRight (C) 2003-2020, Jose M. Martin-Garcia, under the General Public License.
Connecting to external linux executable...
Connection established.
------------------------------------------------------------
Package xAct`xTensor`  version 1.2.0, {2021, 10, 17}
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------------------------------------------------------------
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  it under certain conditions. See the General Public License for details.
------------------------------------------------------------
------------------------------------------------------------
Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}
CopyRight (C) 2003-2020, Jose M. Martin-Garcia, under the General Public License.
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------------------------------------------------------------
Connecting to external linux executable...
Connection established.
------------------------------------------------------------
Package xAct`xPert`  version 1.0.6, {2018, 2, 28}
CopyRight (C) 2005-2020, David Brizuela, Jose M. Martin-Garcia
  and Guillermo A. Mena Marugan, under the General Public License.
** Variable $PrePrint assigned value ScreenDollarIndices
```

```
** Variable $CovDFormat changed from Prefix to Postfix

** Option AllowUpperDerivatives of ContractMetric changed from False to True

** Option MetricOn of MakeRule changed from None to All

** Option ContractMetrics of MakeRule changed from False to True

------------------------------------------------------------

Package xAct`Invar`  version 2.0.5, {2013, 7, 1}

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  D. Yllanes and R. Portugal, under the General Public License.

** DefConstantSymbol: Defining constant symbol sigma.

** DefConstantSymbol: Defining constant symbol dim.

** Option CurvatureRelations of DefCovD changed from True to False

** Variable $CommuteCovDsOnScalars changed from True to False

------------------------------------------------------------

Package xAct`xCoba`  version 0.8.6, {2021, 2, 28}

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  Jose M. Martin-Garcia, under the General Public License.

------------------------------------------------------------

Package xAct`SymManipulator`  version 0.9.5, {2021, 9, 14}

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------------------------------------------------------------

Package xAct`xTras`  version 1.4.2, {2014, 10, 30}

CopyRight (C) 2012–2014, Teake Nutma, under the General Public License.

** Variable $CovDFormat changed from Postfix to Prefix

** Option CurvatureRelations of DefCovD changed from False to True

------------------------------------------------------------

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------------------------------------------------------------

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------------------------------------------------------------
```

```
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  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.
------------------------------------------------------------
** DefManifold: Defining manifold M4.
** DefVBundle: Defining vbundle TangentM4.
** DefTensor: Defining symmetric metric tensor G[-a, -b].
** DefTensor: Defining antisymmetric tensor epsilonG[-a, -a1, -b, -b1].
** DefTensor: Defining tetrametric TetraG[-a, -a1, -b, -b1].
** DefTensor: Defining tetrametric TetraG†[-a, -a1, -b, -b1].
** DefCovD: Defining covariant derivative CD[-a].
** DefTensor: Defining vanishing torsion tensor TorsionCD[a, -a1, -b].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[a, -a1, -b].
** DefTensor: Defining Riemann tensor RiemannCD[-a, -a1, -b, -b1].
** DefTensor: Defining symmetric Ricci tensor RicciCD[-a, -a1].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-a, -a1].
** DefTensor: Defining Weyl tensor WeylCD[-a, -a1, -b, -b1].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-a, -a1].
** DefTensor: Defining Kretschmann scalar KretschmannCD[].
** DefCovD:  Computing RiemannToWeylRules for dim 4
** DefCovD:  Computing RicciToTFRicci for dim 4
** DefCovD:  Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining symmetrized Riemann tensor SymRiemannCD[-a, -a1, -b, -b1].
** DefTensor: Defining symmetric Schouten tensor SchoutenCD[-a, -a1].
** DefTensor: Defining symmetric cosmological Schouten tensor SchoutenCCCD[LI[_], -a, -a1].
** DefTensor: Defining symmetric cosmological Einstein tensor EinsteinCCCD[LI[_], -a, -a1].
** DefCovD: Defining covariant derivative CD[-a]. to be symmetrizable
** DefTensor: Defining weight +2 density DetG[]. Determinant.
** DefParameter: Defining parameter PerturbationParameterG.
** DefTensor: Defining tensor PerturbationG[LI[order], -a, -a1].
```

Define a Planck mass.

** DefConstantSymbol: Defining constant symbol MPl.

Define some dimensionless cosmological couplings.

** DefConstantSymbol: Defining constant symbol Ups1.

** DefConstantSymbol: Defining constant symbol Ups2.

** DefConstantSymbol: Defining constant symbol Alp0.

** DefConstantSymbol: Defining constant symbol Sig1.

** DefConstantSymbol: Defining constant symbol Sig2.

** DefConstantSymbol: Defining constant symbol Sig3.

We will define the conditions which restrict to the theory of interest. This theory is considered in arXiv:2006.03581, and corresponds to Case 2 in arXiv:1910.14197. It is an extension of the theory considered in arXiv:2003.02690 (which itself corresponds to Case 16 in arXiv:1910.14197), because it allows for nonzero value of υ_1. The effect of this extension is to introduce an emergent cosmological constant into the theory, as we will see.

$$\left\{\alpha_0 \to 0,\ \sigma_3 \to 0,\ \sigma_2 \to \sigma_1,\ \upsilon_2 \to -\frac{4}{3}\right\}$$

Define the scale factor.

** DefScalarFunction: Defining scalar function Sf.

** DefScalarFunction: Defining scalar function H.

Define the chart for spherical polar coordinates.

** DefChart: Defining chart SphericalPolar.

** DefTensor: Defining coordinate scalar ct[].

** DefTensor: Defining coordinate scalar cr[].

** DefTensor: Defining coordinate scalar ctheta[].

** DefTensor: Defining coordinate scalar cphi[].

** DefMapping: Defining mapping SphericalPolar.

** DefMapping: Defining inverse mapping iSphericalPolar.

** DefTensor: Defining mapping differential tensor diSphericalPolar[-a, iSphericalPolara].

** DefTensor: Defining mapping differential tensor dSphericalPolar[-a, SphericalPolara].

** DefBasis: Defining basis SphericalPolar. Coordinated basis.

** DefCovD: Defining parallel derivative PDSphericalPolar[-a].

** DefTensor: Defining vanishing torsion tensor TorsionPDSphericalPolar[a, -a1, -b].

** DefTensor: Defining symmetric Christoffel tensor
ChristoffelPDSphericalPolar[a, -a1, -b].

** DefTensor: Defining vanishing Riemann tensor RiemannPDSphericalPolar[-a, -a1, -b, b1].

** DefTensor: Defining vanishing Ricci tensor RicciPDSphericalPolar[-a, -a1].

** DefTensor: Defining antisymmetric +1 density etaUpSphericalPolar[a, a1, b, b1].

** DefTensor: Defining antisymmetric -1 density etaDownSphericalPolar[-a, -a1, -b, -b1].

Set the components of the metric to those of flat FLRW.

Added independent rule $\mathring{g}_{00} \to 1$ for tensor G

Added independent rule $\mathring{g}_{01} \to 0$ for tensor G

Added independent rule $\mathring{g}_{02} \to 0$ for tensor G

Added independent rule $\mathring{g}_{03} \to 0$ for tensor G

Added dependent rule $\mathring{g}_{10} \to \mathring{g}_{01}$ for tensor G

Added independent rule $\mathring{g}_{11} \to -a[t]^2$ for tensor G

Added independent rule $\mathring{g}_{12} \to 0$ for tensor G

Added independent rule $\mathring{g}_{13} \to 0$ for tensor G

Added dependent rule $\mathring{g}_{20} \to \mathring{g}_{02}$ for tensor G

Added dependent rule $\mathring{g}_{21} \to \mathring{g}_{12}$ for tensor G

Added independent rule $\mathring{g}_{22} \to -r^2\, a[t]^2$ for tensor G

Added independent rule $\mathring{g}_{23} \to 0$ for tensor G

Added dependent rule $\mathring{g}_{30} \to \mathring{g}_{03}$ for tensor G

Added dependent rule $\mathring{g}_{31} \to \mathring{g}_{13}$ for tensor G

Added dependent rule $\mathring{g}_{32} \to \mathring{g}_{23}$ for tensor G

Added independent rule $\mathring{g}_{33} \to -r^2\, a[t]^2\, \mathrm{Sin}[\theta]^2$ for tensor G

** DefTensor: Defining weight +2 density DetGSphericalPolar[]. Determinant.

** DefTensor: Defining tensor ChristoffelCDPDSphericalPolar[a, -a1, -b].

Obtaining the field equations from the minisuperspace Lagrangian which comes from the full Poincaré gauge theory

Define scalar functions of the time coordinate scalar, and functions which will represent the ADM quantities.

** DefScalarFunction: Defining scalar function Psis.

** DefScalarFunction: Defining scalar function Phis.

** DefScalarFunction: Defining scalar function Vs.

** DefScalarFunction: Defining scalar function Us.

The minisuperspace Lagrangian, so as to confirm that in the original (physical) Jordan conformal frame this corresponds to the coordinate-imposed metric analogue. As defined in (12) from the paper, we have after an exercise in data entry and double-checking:

$$
\begin{aligned}
&\frac{3}{4}\, v[t] \Big(2\,\sigma_3\,\phi[t]^4\,u[t]^4\,v[t]^2 + 2\,\sigma_3\,\psi[t]^4\,u[t]^4\,v[t]^2 - 8\,\sigma_1\,\phi[t]^3\,u[t]^2\,v[t]\,\big(v[t]\,u'[t] + u[t]\,v'[t]\big) + \\
&\quad 4\,\phi[t]\,u[t]\,v[t]\,\Big(\big(\mathcal{M}_{\mathrm{Pl}}{}^2\,(\alpha_0 + \upsilon_2) + 2\,\sigma_1\,\psi[t]^2\big)\,u[t] + 4\,\sigma_3\,\phi'[t]\Big)\,\big(v[t]\,u'[t] + u[t]\,v'[t]\big) - \\
&\quad 8\,(\sigma_2 - \sigma_3)\,\psi[t]\,u[t]\,v[t]\,\psi'[t]\,\big(v[t]\,u'[t] + u[t]\,v'[t]\big) + \phi[t]^2 \\
&\quad\ \Big(\big(\mathcal{M}_{\mathrm{Pl}}{}^2\,(\alpha_0 + \upsilon_2) - 4\,\sigma_2\,\psi[t]^2\big)\,u[t]^4\,v[t]^2 - 8\,\sigma_1\,u[t]^3\,v[t]^2\,\phi'[t] + 8\,\sigma_3\,v[t]^2\,u'[t]^2 + 16\,\sigma_3\,u[t]\,v[t]\,u'[t]\,v'[t] + 8\,\sigma_3\,u[t]^2\,v'[t]^2\Big) + \\
&\quad \psi[t]^2\,\Big(-\mathcal{M}_{\mathrm{Pl}}{}^2\,\big(\alpha_0 - 4\,\upsilon_1\big)\,u[t]^4\,v[t]^2 + 8\,\sigma_1\,u[t]^3\,v[t]^2\,\phi'[t] + 4\,(-\sigma_2 + \sigma_3)\,v[t]^2\,u'[t]^2 + \\
&\qquad\quad 8\,(-\sigma_2 + \sigma_3)\,u[t]\,v[t]\,u'[t]\,v'[t] + 4\,(-\sigma_2 + \sigma_3)\,u[t]^2\,v'[t]^2\Big) + \\
&\quad 4\,\Big(\mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_2\,v[t]^2\,u'[t]^2 + 2\,\mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_2\,u[t]\,v[t]\,u'[t]\,v'[t] + u[t]^2\,\Big(v[t]^2\,\big(2\,\sigma_3\,\phi'[t]^2 + (-\sigma_2 + \sigma_3)\,\psi'[t]^2\big) + \mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_2\,v'[t]^2\Big)\Big)\Big)
\end{aligned}
$$

I'd like to note here a typo in 2006.03581, in that Eq. (12) has the notation to suggest that it would need an extra factor of $u^{4}v$^3 in order to be gauge covariant (i.e. the measure in flat FLRW spacetime). No such factor is required, so it is safe to take variations directly from the quantity written in that equation.

Mathematica provides some very rudimentary functional tools. We take the Euler–Lagrange equations in the limit of u and v post-variations as described in the paper, and simplify to expressions involving only the Hubble number and the scalar fields assuming only that the scale factor never vanishes.

The u equation:

$$
\begin{aligned}
&2\,\sigma_3\,\phi[t]^4 - \alpha_0\,\mathcal{M}_{\mathrm{Pl}}{}^2\,\psi[t]^2 + 4\,\mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_1\,\psi[t]^2 + 2\,\sigma_3\,\psi[t]^4 - \\
&\ 4\,\mathcal{H}[t]^2\,\big(\mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_2 + 2\,\sigma_3\,\phi[t]^2 + (-\sigma_2 + \sigma_3)\,\psi[t]^2\big) - 2\,\mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_2\,\mathcal{H}'[t] + 2\,\sigma_2\,\psi[t]^2\,\mathcal{H}'[t] - 2\,\sigma_3\,\psi[t]^2\,\mathcal{H}'[t] + \\
&\ \phi[t]^2\,\big(\mathcal{M}_{\mathrm{Pl}}{}^2\,(\alpha_0 + \upsilon_2) - 4\,\sigma_2\,\psi[t]^2 - 4\,\sigma_3\,\mathcal{H}'[t]\big) - \alpha_0\,\mathcal{M}_{\mathrm{Pl}}{}^2\,\phi'[t] - \mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_2\,\phi'[t] + 4\,\sigma_1\,\psi[t]^2\,\phi'[t] - \\
&\ 6\,\mathcal{H}[t]\,\big(2\,\sigma_3\,\phi[t]\,\phi'[t] + (-\sigma_2 + \sigma_3)\,\psi[t]\,\psi'[t]\big) - 4\,\phi[t]\,\big(\sigma_1\,\psi[t]\,\psi'[t] + \sigma_3\,\phi''[t]\big) + 2\,\sigma_2\,\psi[t]\,\psi''[t] - 2\,\sigma_3\,\psi[t]\,\psi''[t]
\end{aligned}
$$

The v equation:

$$
\begin{aligned}
&6\,\sigma_3\,\phi[t]^4 - 3\,\alpha_0\,\mathcal{M}_{\mathrm{Pl}}{}^2\,\psi[t]^2 + 12\,\mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_1\,\psi[t]^2 + 6\,\sigma_3\,\psi[t]^4 - 12\,\mathcal{H}[t]^2\,\big(\mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_2 + 2\,\sigma_3\,\phi[t]^2 + (-\sigma_2 + \sigma_3)\,\psi[t]^2\big) - \\
&\ 8\,\mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_2\,\mathcal{H}'[t] + 8\,\sigma_2\,\psi[t]^2\,\mathcal{H}'[t] - 8\,\sigma_3\,\psi[t]^2\,\mathcal{H}'[t] + \phi[t]^2\,\big(3\,\mathcal{M}_{\mathrm{Pl}}{}^2\,(\alpha_0 + \upsilon_2) - 12\,\sigma_2\,\psi[t]^2 - 16\,\sigma_3\,\mathcal{H}'[t]\big) - \\
&\ 4\,\alpha_0\,\mathcal{M}_{\mathrm{Pl}}{}^2\,\phi'[t] - 4\,\mathcal{M}_{\mathrm{Pl}}{}^2\,\upsilon_2\,\phi'[t] + 16\,\sigma_1\,\psi[t]^2\,\phi'[t] + 8\,\sigma_3\,\phi'[t]^2 - 4\,\sigma_2\,\psi'[t]^2 + 4\,\sigma_3\,\psi'[t]^2 - \\
&\ 16\,\mathcal{H}[t]\,\big(2\,\sigma_3\,\phi[t]\,\phi'[t] + (-\sigma_2 + \sigma_3)\,\psi[t]\,\psi'[t]\big) - 16\,\phi[t]\,\big(\sigma_1\,\psi[t]\,\psi'[t] + \sigma_3\,\phi''[t]\big) + 8\,\sigma_2\,\psi[t]\,\psi''[t] - 8\,\sigma_3\,\psi[t]\,\psi''[t]
\end{aligned}
$$

The ϕ equation:

$$-16\,\sigma_3\,\mathcal{H}[t]^2\,\phi[t]+4\,\sigma_3\,\phi[t]^3+\phi[t]\left(\mathcal{M}_{\text{Pl}}{}^2\,(\alpha_0+\upsilon_2)-4\,\sigma_2\,\psi[t]^2-8\,\sigma_3\,\mathcal{H}'[t]\right)+$$
$$2\,\mathcal{H}[t]\left(\mathcal{M}_{\text{Pl}}{}^2\,(\alpha_0+\upsilon_2)-4\,\sigma_1\,\psi[t]^2-12\,\sigma_3\,\phi'[t]\right)-8\left(\sigma_1\,\psi[t]\,\psi'[t]+\sigma_3\,\phi''[t]\right)$$

The ψ equation:

$$4\,\sigma_3\,\psi[t]^3+\psi[t]\left(-\alpha_0\,\mathcal{M}_{\text{Pl}}{}^2+4\,\mathcal{M}_{\text{Pl}}{}^2\,\upsilon_1+8\,(\sigma_2-\sigma_3)\,\mathcal{H}[t]^2+8\,\sigma_1\,\mathcal{H}[t]\,\phi[t]-4\,\sigma_2\,\phi[t]^2+4\,\sigma_2\,\mathcal{H}'[t]-4\,\sigma_3\,\mathcal{H}'[t]+8\,\sigma_1\,\phi'[t]\right)+$$
$$4\,(\sigma_2-\sigma_3)\left(3\,\mathcal{H}[t]\,\psi'[t]+\psi''[t]\right)$$

We will next move over to the scalar-tensor theory.

Obtaining the field equations from the metrical analogue in the (physical) Jordan frame

Define the scalar functions for the Jordan conformal frame. These are defined in Eq. (9) of the paper.

** DefTensor: Defining tensor Phi[].

** DefTensor: Defining tensor Psi[].

Define a scalar function which denotes Mathematica's square root Sqrt.

** DefScalarFunction: Defining scalar function Sq.

Define also a constant to manually trace the sign of the time component of the vector field. The need for this (implicit) constant is referred to at the top of Section VI, and I'm hoping that it could be one of the sources of error in the unusual-looking potential.

** DefConstantSymbol: Defining constant symbol sgn.

Define the current in (14b), note that the valence of the current is important because it is really a bunch of derivatives.

** DefTensor: Defining tensor J[-a].

$$\mathcal{J}^{\alpha}$$

$$-\alpha_0\,\mathcal{M}_{\text{Pl}}{}^2\left(\mathring{\nabla}_{\alpha}\phi\right)-\mathcal{M}_{\text{Pl}}{}^2\,\upsilon_2\left(\mathring{\nabla}_{\alpha}\phi\right)+4\,\sigma_1\,\psi^2\left(\mathring{\nabla}_{\alpha}\phi\right)-4\,\sigma_1\,\phi\,\psi\left(\mathring{\nabla}_{\alpha}\psi\right)$$

Take a moment to remind ourselves how it comes to pass that the Cuscuton does not contribute any energy density using this Lagrangian.

$$\sqrt{-\overset{\approx}{\mathring{g}}}\ \sqrt{\mathcal{J}_{\alpha}\,\mathcal{J}^{\alpha}}$$

Here are the metric field equations from the density above. So we can see that the time-time component will vanish.

$$-\frac{1}{2}\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\ \overset{\circ}{g}_{\alpha\beta}\ \sqrt{(\mathcal{T}_{\alpha}\ \mathcal{T}^{\alpha})}+\frac{\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\ \mathcal{T}_{\alpha}\ \mathcal{T}_{\beta}\ \sqrt{(\mathcal{T}_{\alpha}\ \mathcal{T}^{\alpha})}}{2\,(\mathcal{T}_{\text{a\$9997}})\,(\mathcal{T}^{\text{a\$9997}})}$$

Those last two outputs were just a brief tangent, so now we can carry on.

Define the general Jordan frame metrical analogue in (14a), remembering that the square root is now accommodated by an explicit sign and formal function.

$$\frac{1}{4}\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\Big(3\,\alpha_0\,\mathcal{M}_{\text{Pl}}^2\,\phi^2+3\,\mathcal{M}_{\text{Pl}}^2\,\upsilon_2\,\phi^2+6\,\sigma_3\,\phi^4-3\,\alpha_0\,\mathcal{M}_{\text{Pl}}^2\,\psi^2+12\,\mathcal{M}_{\text{Pl}}^2\,\upsilon_1\,\psi^2-12\,\sigma_2\,\phi^2\,\psi^2+6\,\sigma_3\,\psi^4+2\,\mathcal{M}_{\text{Pl}}^2\,\upsilon_2\,\text{R}[\overset{\circ}{\nabla}]+4\,\sigma_3\,\phi^2\,\text{R}[\overset{\circ}{\nabla}]-2\,\sigma_2\,\psi^2\,\text{R}[\overset{\circ}{\nabla}]+2\,\sigma_3\,\psi^2\,\text{R}[\overset{\circ}{\nabla}]+4\,s\,\mathcal{Q}[(\mathcal{T}_{\alpha}\ \mathcal{T}^{\alpha})]+24\,\sigma_3\,(\overset{\circ}{\nabla}_{\alpha}\phi)(\overset{\circ}{\nabla}{}^{\alpha}\phi)-12\,\sigma_2\,(\overset{\circ}{\nabla}_{\alpha}\psi)(\overset{\circ}{\nabla}{}^{\alpha}\psi)+12\,\sigma_3\,(\overset{\circ}{\nabla}_{\alpha}\psi)(\overset{\circ}{\nabla}{}^{\alpha}\psi)\Big)$$

The Jordan frame metrical analogue with the constraints imposed on the couplings.

$$\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\Bigg(-3\,\sigma_1\,\phi^2\,\psi^2+\frac{3}{4}\,\mathcal{M}_{\text{Pl}}^2\left(-\frac{4\,\phi^2}{3}+4\,\upsilon_1\,\psi^2\right)+\left(-\frac{2\,\mathcal{M}_{\text{Pl}}^2}{3}-\frac{\sigma_1\,\psi^2}{2}\right)\text{R}[\overset{\circ}{\nabla}]+s\,\mathcal{Q}\Big[\Big(\frac{4}{3}\,\mathcal{M}_{\text{Pl}}^2\,(\overset{\circ}{\nabla}_{\alpha}\phi)+4\,\sigma_1\,\psi^2\,(\overset{\circ}{\nabla}_{\alpha}\phi)-4\,\sigma_1\,\phi\,\psi\,(\overset{\circ}{\nabla}_{\alpha}\psi)\Big)\Big(\frac{4}{3}\,\mathcal{M}_{\text{Pl}}^2\,(\overset{\circ}{\nabla}{}^{\alpha}\phi)+4\,\sigma_1\,\psi^2\,(\overset{\circ}{\nabla}{}^{\alpha}\phi)-4\,\sigma_1\,\phi\,\psi\,(\overset{\circ}{\nabla}{}^{\alpha}\psi)\Big)\Big]-3\,\sigma_1\,(\overset{\circ}{\nabla}_{\alpha}\psi)(\overset{\circ}{\nabla}{}^{\alpha}\psi)\Bigg)$$

This square root is a pattern which will stubbornly remain in the field equations when we make no assumptions. Since we deal only in real quantities, we know (Mathematica does not) that it is equal to the absolute value of the quantity inside the square, or the absolute value of the timelike part of the vector.

$$\sqrt{\left((\mathcal{M}_{\text{Pl}}^2+3\,\sigma_1\,\psi[t]^2)\,\phi'[t]-3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2}$$

This is the timelike part of the vector.

$$(\mathcal{M}_{\text{Pl}}^2+3\,\sigma_1\,\psi[t]^2)\,\phi'[t]-3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]$$

This means that we can safely impose the following replacement rule.

$$\left\{\sqrt{\left((\mathcal{M}_{\text{Pl}}^2+3\,\sigma_1\,\psi[t]^2)\,\phi'[t]-3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2}\rightarrow\frac{(\mathcal{M}_{\text{Pl}}^2+3\,\sigma_1\,\psi[t]^2)\,\phi'[t]-3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]}{s}\right\}$$

Using the more sophisticated tools out of xAct, we can provide covariant expressions for the components of the field equations from the metrical analogue in the Jordan frame.

The Einstein field equations.

$$\frac{1}{2}\,\mathcal{M}_{\text{Pl}}^2\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\ \overset{\circ}{g}_{\mu\nu}\ \phi^2-\frac{3}{2}\,\mathcal{M}_{\text{Pl}}^2\,\upsilon_1\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\ \overset{\circ}{g}_{\mu\nu}\ \psi^2+\frac{3}{2}\,\sigma_1\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\ \overset{\circ}{g}_{\mu\nu}\ \phi^2\,\psi^2-\frac{2}{3}\,\mathcal{M}_{\text{Pl}}^2\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\ \text{R}[\overset{\circ}{\nabla}]_{\mu\nu}-\frac{1}{2}\,\sigma_1\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\ \psi^2\ \text{R}[\overset{\circ}{\nabla}]_{\mu\nu}+\frac{1}{3}\,\mathcal{M}_{\text{Pl}}^2\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\ \overset{\circ}{g}_{\mu\nu}\ \text{R}[\overset{\circ}{\nabla}]+\frac{1}{4}\,\sigma_1\sqrt{-\overset{\approx}{\overset{\circ}{g}}}\ \overset{\circ}{g}_{\mu\nu}\ \psi^2\ \text{R}[\overset{\circ}{\nabla}]-$$

$$\frac{1}{2}\, s\, \sqrt{-\mathring{\tilde{\tilde{g}}}}\; \mathring{g}_{\mu\nu}\; Q\Big[\frac{16}{9}\mathcal{M}_{\text{Pl}}^4 \left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\psi^2\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\psi^4\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\phi\psi\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - 32\sigma_1^2\phi\psi^3\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\phi^2\psi^2\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\psi\right)\right)\Big] -$$

$$\sigma_1\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\mathring{g}_{\mu\nu}\;\psi\left(\mathring{\nabla}_\alpha\mathring{\nabla}^\alpha\psi\right) + \frac{1}{2}\sigma_1\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\mathring{g}_{\mu\nu}\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\psi\right) + \frac{1}{2}\sigma_1\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\psi\left(\mathring{\nabla}_\mu\mathring{\nabla}_\nu\psi\right) -$$

$$2\sigma_1\sqrt{-\mathring{\tilde{\tilde{g}}}}\left(\mathring{\nabla}_\mu\psi\right)\left(\mathring{\nabla}_\nu\psi\right) + \frac{1}{2}\sigma_1\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\psi\left(\mathring{\nabla}_\nu\mathring{\nabla}_\mu\psi\right) +$$

$$\frac{16}{9}\mathcal{M}_{\text{Pl}}^4 s\sqrt{-\mathring{\tilde{\tilde{g}}}}\left(\mathring{\nabla}_\mu\phi\right)\left(\mathring{\nabla}_\nu\phi\right) Q'\Big[\frac{16}{9}\mathcal{M}_{\text{Pl}}^4 \left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\psi^2\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\psi^4\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\phi\psi\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - 32\sigma_1^2\phi\psi^3\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\phi^2\psi^2\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\psi\right)\right)\Big] + \frac{32}{3}\mathcal{M}_{\text{Pl}}^2 s\sigma_1\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\psi^2\left(\mathring{\nabla}_\mu\phi\right)\left(\mathring{\nabla}_\nu\phi\right) Q'\Big[\frac{16}{9}\mathcal{M}_{\text{Pl}}^4 \left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\psi^2\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\psi^4\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\phi\psi\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - 32\sigma_1^2\phi\psi^3\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\phi^2\psi^2\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\psi\right)\right)\Big] +$$

$$16 s\sigma_1^2\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\psi^4\left(\mathring{\nabla}_\mu\phi\right)\left(\mathring{\nabla}_\nu\phi\right) Q'\Big[\frac{16}{9}\mathcal{M}_{\text{Pl}}^4 \left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\psi^2\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\psi^4\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\phi\psi\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - 32\sigma_1^2\phi\psi^3\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\phi^2\psi^2\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\psi\right)\right)\Big] - \frac{16}{3}\mathcal{M}_{\text{Pl}}^2 s\sigma_1\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\phi\psi\left(\mathring{\nabla}_\mu\psi\right)\left(\mathring{\nabla}_\nu\phi\right) Q'\Big[\frac{16}{9}\mathcal{M}_{\text{Pl}}^4 \left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\psi^2\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\psi^4\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\phi\psi\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - 32\sigma_1^2\phi\psi^3\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\phi^2\psi^2\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\psi\right)\right)\Big] -$$

$$16 s\sigma_1^2\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\phi\psi^3\left(\mathring{\nabla}_\mu\psi\right)\left(\mathring{\nabla}_\nu\phi\right) Q'\Big[\frac{16}{9}\mathcal{M}_{\text{Pl}}^4 \left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\psi^2\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\psi^4\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\phi\psi\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - 32\sigma_1^2\phi\psi^3\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\phi^2\psi^2\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\psi\right)\right)\Big] - \frac{16}{3}\mathcal{M}_{\text{Pl}}^2 s\sigma_1\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\phi\psi\left(\mathring{\nabla}_\mu\phi\right)\left(\mathring{\nabla}_\nu\psi\right) Q'\Big[\frac{16}{9}\mathcal{M}_{\text{Pl}}^4 \left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\psi^2\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\psi^4\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\phi\psi\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - 32\sigma_1^2\phi\psi^3\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\phi^2\psi^2\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\psi\right)\right)\Big] -$$

$$16 s\sigma_1^2\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\phi\psi^3\left(\mathring{\nabla}_\mu\phi\right)\left(\mathring{\nabla}_\nu\psi\right) Q'\Big[\frac{16}{9}\mathcal{M}_{\text{Pl}}^4 \left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\psi^2\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\psi^4\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\phi\psi\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - 32\sigma_1^2\phi\psi^3\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\phi^2\psi^2\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\psi\right)\right)\Big] + 16 s\sigma_1^2\sqrt{-\mathring{\tilde{\tilde{g}}}}\;\phi^2\psi^2\left(\mathring{\nabla}_\mu\psi\right)\left(\mathring{\nabla}_\nu\psi\right) Q'\Big[\frac{16}{9}\mathcal{M}_{\text{Pl}}^4 \left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\psi^2\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\psi^4\left(\left(\mathring{\nabla}_\alpha\phi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - \frac{32}{3}\mathcal{M}_{\text{Pl}}^2\sigma_1\phi\psi\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) - 32\sigma_1^2\phi\psi^3\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\phi\right)\right) + 16\sigma_1^2\phi^2\psi^2\left(\left(\mathring{\nabla}_\alpha\psi\right)\left(\mathring{\nabla}^\alpha\psi\right)\right)\Big]$$

The expressions for the ϕ and ψ equations
are somewhat lengthy, but they are likewise covariant.

With some aspects of xCoba, we can also
transfer these equations into the background FLRW equations.

The constraint (density) equation.

$$-r^2 a[t]^8 \left(-\phi[t]^2 \left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right) + \mathcal{H}[t]^2 \left(4\,\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right) + 6\,\sigma_1\,\mathcal{H}[t]\,\psi[t]\,\psi'[t] + 3\left(\mathcal{M}_{\text{Pl}}{}^2\,\upsilon_1\,\psi[t]^2 + \sigma_1\,\psi'[t]^2\right)\right)$$

The dynamical (pressure) equation.

$$\sqrt{r^4 a[t]^6}\,\Big(9\,\mathcal{M}_{\text{Pl}}{}^2\,\upsilon_1\,\psi[t]^2 - 3\,\phi[t]^2\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right) + 3\,\mathcal{H}[t]^2\left(4\,\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right) + 8\,\mathcal{M}_{\text{Pl}}{}^2\,\mathcal{H}'[t] + 6\,\sigma_1\,\psi[t]^2\,\mathcal{H}'[t] + 12\,\sigma_1\,\mathcal{H}[t]\,\psi[t]\,\psi'[t] - 3\,\sigma_1\,\psi'[t]^2 + 4\,s\,\sqrt{\left(\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2} + 6\,\sigma_1\,\psi[t]\,\psi''[t]\Big)$$

The ϕ equation.

$$-r^2 a[t]^8 \Big(6\,\mathcal{M}_{\text{Pl}}{}^2\,s\,\sigma_1\,\psi[t]\,\phi'[t]\,\psi'[t] + 18\,s\,\sigma_1{}^2\,\psi[t]^3\,\phi'[t]\,\psi'[t] + 2\,s\,\mathcal{H}[t]\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\left(\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right) + \mathcal{M}_{\text{Pl}}{}^2\,\phi[t]\,\sqrt{\left(\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2} + 3\,\sigma_1\,\phi[t]\,\psi[t]^2\left(-6\,s\,\sigma_1\,\psi'[t]^2 + \sqrt{\left(\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2}\right)\Big)$$

The ψ equation.

$$\begin{aligned} r^2 a[t]^8 \Big(&2\,\mathcal{M}_{\text{Pl}}{}^2\,s\,\sigma_1\,\mathcal{H}[t]\,\phi[t]\,\psi[t]\,\phi'[t] + 6\,s\,\sigma_1{}^2\,\mathcal{H}[t]\,\phi[t]\,\psi[t]^3\,\phi'[t] + \\ &2\,\mathcal{M}_{\text{Pl}}{}^2\,s\,\sigma_1\,\psi[t]\,\phi'[t]^2 + 6\,s\,\sigma_1{}^2\,\psi[t]^3\,\phi'[t]^2 - 6\,s\,\sigma_1{}^2\,\mathcal{H}[t]\,\phi[t]^2\,\psi[t]^2\,\psi'[t] - \\ &6\,s\,\sigma_1{}^2\,\phi[t]\,\psi[t]^2\,\phi'[t]\,\psi'[t] + \mathcal{M}_{\text{Pl}}{}^2\,\upsilon_1\,\psi[t]\,\sqrt{\left(\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2} + \\ &2\,\sigma_1\,\mathcal{H}[t]^2\,\psi[t]\,\sqrt{\left(\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2} - \\ &\sigma_1\,\phi[t]^2\,\psi[t]\,\sqrt{\left(\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2} + \\ &\sigma_1\,\psi[t]\,\mathcal{H}'[t]\,\sqrt{\left(\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2} + \\ &3\,\sigma_1\,\mathcal{H}[t]\,\psi'[t]\,\sqrt{\left(\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2} + \\ &\sigma_1\,\sqrt{\left(\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)^2}\,\psi''[t]\Big) \end{aligned}$$

Now finally we want to be able to remove the square
root, using the replacement rule defined above. It is important
here to check that there is no dependence on s in the final answer.

The constraint (density) equation.

$$-r^2 a[t]^8 \left(-\phi[t]^2 \left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right) + \mathcal{H}[t]^2 \left(4\,\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right) + 6\,\sigma_1\,\mathcal{H}[t]\,\psi[t]\,\psi'[t] + 3\left(\mathcal{M}_{\text{Pl}}{}^2\,\upsilon_1\,\psi[t]^2 + \sigma_1\,\psi'[t]^2\right)\right)$$

The dynamical (pressure) equation.

$$\sqrt{r^4 a[t]^6}\,\Big(9\,\mathcal{M}_{\text{Pl}}{}^2\,\upsilon_1\,\psi[t]^2 - 3\,\phi[t]^2\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right) + 3\,\mathcal{H}[t]^2\left(4\,\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right) + 8\,\mathcal{M}_{\text{Pl}}{}^2\,\mathcal{H}'[t] + 6\,\sigma_1\,\psi[t]^2\,\mathcal{H}'[t] + 4\left(\mathcal{M}_{\text{Pl}}{}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] + 12\,\sigma_1\,\mathcal{H}[t]\,\psi[t]\,\psi'[t] - 12\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t] - 3\,\sigma_1\,\psi'[t]^2 + 6\,\sigma_1\,\psi[t]\,\psi''[t]\Big)$$

The ϕ equation.

$$-r^2\, a[t]^8 \left(2\,\mathcal{H}[t] \left(\mathcal{M}_{\mathrm{Pl}}^2 + 3\,\sigma_1\,\psi[t]^2\right) + \phi[t] \left(\mathcal{M}_{\mathrm{Pl}}^2 + 3\,\sigma_1\,\psi[t]^2\right) + 6\,\sigma_1\,\psi[t]\,\psi'[t]\right) \left(\left(\mathcal{M}_{\mathrm{Pl}}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right)$$

The ψ equation.

$$r^2\, a[t]^8 \left(\left(\mathcal{M}_{\mathrm{Pl}}^2 + 3\,\sigma_1\,\psi[t]^2\right)\phi'[t] - 3\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t]\right) \left(\psi[t] \left(\mathcal{M}_{\mathrm{Pl}}^2\,\upsilon_1 + 2\,\sigma_1\,\mathcal{H}[t]^2 + 2\,\sigma_1\,\mathcal{H}[t]\,\phi[t] - \sigma_1\,\phi[t]^2 + \sigma_1\,\mathcal{H}'[t] + 2\,\sigma_1\,\phi'[t]\right) + \sigma_1 \left(3\,\mathcal{H}[t]\,\psi'[t] + \psi''[t]\right)\right)$$

We've obtained the scalar-tensor field equations, and some straightforward checks will be enough for us to show that the minisuperspace equations are precisely the same. We won't do these checks here, but instead continue with the analysis and try to find the details of the background solutions that define the torsion condensate.

Let's work from the minisuperspace equations which we obtained earlier.

The u equation.

$$\frac{2}{3} \left(6\,\mathcal{M}_{\mathrm{Pl}}^2\,\upsilon_1\,\psi[t]^2 - 2\,\phi[t]^2 \left(\mathcal{M}_{\mathrm{Pl}}^2 + 3\,\sigma_1\,\psi[t]^2\right) + \mathcal{H}[t]^2 \left(8\,\mathcal{M}_{\mathrm{Pl}}^2 + 6\,\sigma_1\,\psi[t]^2\right) + 4\,\mathcal{M}_{\mathrm{Pl}}^2\,\mathcal{H}'[t] + 3\,\sigma_1\,\psi[t]^2\,\mathcal{H}'[t] + 2\,\mathcal{M}_{\mathrm{Pl}}^2\,\phi'[t] + 6\,\sigma_1\,\psi[t]^2\,\phi'[t] + 9\,\sigma_1\,\mathcal{H}[t]\,\psi[t]\,\psi'[t] - 6\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t] + 3\,\sigma_1\,\psi[t]\,\psi''[t]\right)$$

The v equation.

$$\frac{4}{3} \left(9\,\mathcal{M}_{\mathrm{Pl}}^2\,\upsilon_1\,\psi[t]^2 - 3\,\phi[t]^2 \left(\mathcal{M}_{\mathrm{Pl}}^2 + 3\,\sigma_1\,\psi[t]^2\right) + 3\,\mathcal{H}[t]^2 \left(4\,\mathcal{M}_{\mathrm{Pl}}^2 + 3\,\sigma_1\,\psi[t]^2\right) + 8\,\mathcal{M}_{\mathrm{Pl}}^2\,\mathcal{H}'[t] + 6\,\sigma_1\,\psi[t]^2\,\mathcal{H}'[t] + 4\,\mathcal{M}_{\mathrm{Pl}}^2\,\phi'[t] + 12\,\sigma_1\,\psi[t]^2\,\phi'[t] + 12\,\sigma_1\,\mathcal{H}[t]\,\psi[t]\,\psi'[t] - 12\,\sigma_1\,\phi[t]\,\psi[t]\,\psi'[t] - 3\,\sigma_1\,\psi'[t]^2 + 6\,\sigma_1\,\psi[t]\,\psi''[t]\right)$$

The ϕ equation.

$$-\frac{4}{3} \left(2\,\mathcal{H}[t] \left(\mathcal{M}_{\mathrm{Pl}}^2 + 3\,\sigma_1\,\psi[t]^2\right) + \phi[t] \left(\mathcal{M}_{\mathrm{Pl}}^2 + 3\,\sigma_1\,\psi[t]^2\right) + 6\,\sigma_1\,\psi[t]\,\psi'[t]\right)$$

An algebraic solution for the ϕ field based on its field equation.

$$\left\{\left\{\phi[t] \to -\frac{2\left(\mathcal{M}_{\mathrm{Pl}}^2\,\mathcal{H}[t] + 3\,\sigma_1\,\mathcal{H}[t]\,\psi[t]^2 + 3\,\sigma_1\,\psi[t]\,\psi'[t]\right)}{\mathcal{M}_{\mathrm{Pl}}^2 + 3\,\sigma_1\,\psi[t]^2}\right\}\right\}$$

This is the big important line in this script: the rules below define the torsion condensate background solution to the field equations.

$$\left\{\left\{\phi[t] \to \left(1 - \sqrt{3}\right)\mathcal{H}[t],\ \phi'[t] \to \left(1 - \sqrt{3}\right)\mathcal{H}'[t],\ \psi[t] \to \frac{\mathcal{M}_{\mathrm{Pl}}}{\sqrt{3}\,\sqrt{-\sigma_1}},\ \psi'[t] \to 0,\ \psi''[t] \to 0\right\}\right\}$$

This solution is based on the algebraic solution for the ϕ field above. Note that the key feature is that the ψ field goes to a specific constant value. Unhelpfully, the solution for ϕ looks as though it becomes singular at that value, but in fact detailed analysis of the theory in arXiv:2003.02690, arXiv:2006.03581 and some upcoming papers suggests that the (cosmic time) velocity of ψ will vanish faster than the denominator (to see why this is true, you can look at Eq. (116) on page 19 of arXiv:2003.02690, and recall that the torsion condensate or correspondence solution -- same thing -- will be reached early in the matter-dominated epoch such that the fractional difference in ϖ decays like a power law in conformal time). As a result, the ϕ field is just proportional to the Hubble number.

The ψ equation. This is essentially a Klein-Gordon equation.

$$4\,\psi[t]\left(\mathcal{M}_{\text{Pl}}{}^{2}\,\upsilon_1+2\,\sigma_1\,\mathcal{H}[t]^2+2\,\sigma_1\,\mathcal{H}[t]\,\phi[t]-\sigma_1\,\phi[t]^2+\sigma_1\,\mathcal{H}'[t]+2\,\sigma_1\,\phi'[t]\right)+4\,\sigma_1\left(3\,\mathcal{H}[t]\,\psi'[t]+\psi''[t]\right)$$

The combination of the u and v equations in which the derivative of the Hubble number has been eliminated. This corresponds to the time-time Einstein equation, or the Friedmann constraint equation.

$$-\phi[t]^2\left(\mathcal{M}_{\text{Pl}}{}^{2}+3\,\sigma_1\,\psi[t]^2\right)+\mathcal{H}[t]^2\left(4\,\mathcal{M}_{\text{Pl}}{}^{2}+3\,\sigma_1\,\psi[t]^2\right)+6\,\sigma_1\,\mathcal{H}[t]\,\psi[t]\,\psi'[t]+3\left(\mathcal{M}_{\text{Pl}}{}^{2}\,\upsilon_1\,\psi[t]^2+\sigma_1\,\psi'[t]^2\right)$$

Now substituting the torsion condensate background solution into the Friedmann constraint equation.

$$-\frac{\mathcal{M}_{\text{Pl}}{}^{4}\,\upsilon_1}{\sigma_1}+3\,\mathcal{M}_{\text{Pl}}{}^{2}\,\mathcal{H}[t]^2$$

$$-\frac{\mathcal{M}_{\text{Pl}}{}^{4}\,\upsilon_1}{\sigma_1}+3\,\mathcal{M}_{\text{Pl}}{}^{2}\,\mathcal{H}[t]^2$$

We know from arXiv:2006.03581, in the paragraph below Eq. (27), that one of those torsion couplings can be interpreted as an emergent cosmological constant.

** DefConstantSymbol: Defining constant symbol Lamb.

$$-\mathcal{M}_{\text{Pl}}{}^{2}\left(\Lambda-3\,\mathcal{H}[t]^2\right)$$

Now recall that we didn't bother to add any matter in the derivation above. We can re-insert it here in a sneaky way, just by recalling that it must add with the cosmological constant in the usual manner (in order for Λ to have been given this interpretation in the first place).

** DefScalarFunction: Defining scalar function Rhos.

$$-\Lambda\,\mathcal{M}_{\text{Pl}}{}^{2}+3\,\mathcal{M}_{\text{Pl}}{}^{2}\,\mathcal{H}[t]^2-\rho[t]$$

Okay, so this last equation above looks like a healthy statement of the first Friedmann equation. Remember that we obtained this equation from the field equations of the PGT/metrical analogue, and then imposed the torsion condensate/correspondence solution on the values of the fields. The consequence of imposing the correspondence solution is that the background cosmological dynamics are precisely those of GR.

One last thing which would be nice to check, is that the correspondence solution also satisfies the field equation for the ψ field.

$$4\,\mathcal{M}_{\text{Pl}}\left(\mathcal{M}_{\text{Pl}}{}^{2}\,\upsilon_1+\left(3-2\,\sqrt{3}\right)\sigma_1\,\mathcal{H}'[t]\right)$$