$L = L_{\rm PV}\left({}^{(1)}\lambda_{\mu}{}^{\nu\sigma}, {}^{(3)}\lambda^{\mu}, {}^{(2)}\mu, L_{\rm M}\to 0\right)$ w.l.o.g. We package six d.o.f into the 2-form field

$$\begin{aligned} M_{\rm Pl}B_{\mu\nu} \equiv 2\mathring{\nabla}_{\sigma}{}^{(1)}T^{\sigma}{}_{[\mu\nu]} - {}^{(2)}F_{\mu\nu} + \frac{3}{4}\epsilon^{\sigma\lambda}{}_{\mu\nu}{}^{(3)}F_{\sigma\lambda} \\ + 2{}^{(2)}T_{\sigma}{}^{(1)}T^{\sigma}{}_{[\mu\nu]} + 3\epsilon^{\sigma\lambda\rho}{}_{[\mu}{}^{(3)}T_{\sigma}{}^{(1)}T_{\nu]\lambda\rho}, \end{aligned} \tag{4}$$

where the Maxwell field strengths are ${}^{(2)}F_{\mu\nu} \equiv 2\mathring{\nabla}_{[\mu}{}^{(2)}T_{\nu]}$ etc. We introduce Eq. (4) just to simplify the $\omega^{ij}{}_{\mu}$-equations $\delta/\delta\omega^{ij}{}_{\mu}\int \mathrm{d}^4x eL \approx 0$, which decompose to [199]

$${}^{(1)}T^{\sigma}{}_{\mu\nu} \approx \frac{\alpha}{M_{\rm Pl}}\left[\widehat{\mathring{\nabla}B} + \frac{\widehat{B\mathring{\nabla}B}}{M_{\rm Pl}} + \frac{\widehat{B^2\,{}^{(1)}T}}{M_{\rm Pl}}\right], \tag{5a}$$

$${}^{(2)}T_{\mu} \approx \frac{4\alpha}{3M_{\rm Pl}}\left[\mathring{\nabla}_{\nu}B_{\mu}{}^{\nu} - B_{\sigma\lambda}{}^{(1)}T_{\mu}{}^{\sigma\lambda}\right], \tag{5b}$$

$${}^{(3)}T_{\mu} \approx \frac{4\alpha\epsilon_{\mu}{}^{\nu\sigma\lambda}}{9M_{\rm Pl}}\left[\mathring{\nabla}_{\nu}B_{\sigma\lambda} - 2B^{\rho}{}_{\nu}{}^{(1)}T_{\rho\sigma\lambda}\right], \tag{5c}$$

where $\widehat{\cdot\cdot}$ suppresses contractions, but all parts are simplified by (4). With $\alpha \to 0$ we recover entirely vanishing vacuum torsion as expected in EC theory, otherwise Eqs. (5a) to (5c) should be wavelike for dynamical torsion (if any). It is simplest to notice how all torsion dynamics can be confined to $B_{\mu\nu}$, though that variable eliminates a single derivative in Eq. (4). To extract the propagating (second-derivative) equation in $B_{\mu\nu}$, we take the antisymmetrised divergence of (5a), next eliminating $\mathring{\nabla}_{\sigma}{}^{(1)}T^{\sigma}{}_{[\mu\nu]}$ for $B_{\mu\nu}$, ${}^{(1)}T^{\mu}{}_{\nu\sigma}$, ${}^{(2)}T_{\mu}$ and ${}^{(3)}T_{\mu}$ using Eq. (4), then using Eqs. (5b) and (5c) eliminate ${}^{(2)}T_{\mu}$ and ${}^{(3)}T_{\mu}$ for $B_{\mu\nu}$ and ${}^{(1)}T^{\mu}{}_{\nu\sigma}$, before finally recycling Eq. (5a) to eliminate all remaining ${}^{(1)}T^{\mu}{}_{\nu\sigma}$ *perturbatively* in terms of $B_{\mu\nu}$ [199]. Upon integrating, we find (at least on flat space) that the resulting equation descends from the effective theory

$$\begin{aligned} &L_{\rm PV}\left({}^{(1)}\lambda_{\mu}{}^{\nu\sigma}, {}^{(3)}\lambda^{\mu}, {}^{(2)}\mu, L_{\rm M}, \mathring{R}\to 0\right) \\ &\cong -\frac{M_{\rm Pl}{}^2}{2}B_{\mu\nu}B^{\mu\nu} + \frac{\alpha^3}{M_{\rm Pl}{}^2}\widehat{B^2\mathring{\nabla}B\mathring{\nabla}B} + \mathcal{O}\left(B^6\right). \end{aligned} \tag{6}$$

A pathological kinetic term is revealed in (6), not e.g. the safe p-form operator $\mathring{\nabla}_{[\mu}B_{\nu\sigma]}\mathring{\nabla}^{[\mu}B^{\nu\sigma]}$ [200]. Truncate (6) at $\mathcal{O}\left(B^4\right)$ as shown, then heuristically the canonical 2-form has a mass $\sim M_{\rm Pl}{}^2/B$ and becomes *strongly coupled* as $B \to 0$, whereupon it drops out of the linear spectrum. The problem is only aggravated at higher perturbative orders. Next, compare (6) with the seemingly unrelated NGR model [37, 201, 202]. That theory also has a dynamical 2-form $B_{\mu\nu} \equiv M_{\rm Pl}\eta_{ij}\delta^{i}_{[\mu}e^{j}{}_{\nu]}$, which is different to (4). Instead of diverging as $B \to 0$ the NGR 2-form mass *vanishes*, but again this *removes* longitudinal polarisations from the linear spectrum [203–205]. Relative to our Eq. (6), NGR is positioned at the far side of the non-Riemannian landscape: yet both theories fail as minimal deviations from the strict EC or MT models.

IV. Healthy spectrum with multipliers

Obviously Eq. (6) is sick: we will now show that our ${}^{(1)}\lambda_{\mu\nu\sigma}{}^{(1)}T^{\mu\nu\sigma}$ term is the cure. For greater generality we restore the whole axial vector sector by considering $L = L_{\rm PV}\left({}^{(3)}\lambda^{\mu}\to 0\right) + {}^{(3)}\mu M_{\rm Pl}{}^2\,{}^{(3)}T^{\mu}\,{}^{(3)}T_{\mu}$. Using the spin tensor of matter $eS^{\mu}{}_{ij} \equiv -\delta/\delta\omega^{ij}{}_{\mu}\int \mathrm{d}^4x eL_{\rm M}$, $e \equiv \det e^{i}{}_{\mu}$, Eqs. (5a) to (5c) become

$${}^{(1)}S^{\mu}{}_{\nu\sigma} \approx {}^{(1)}\lambda^{\mu}{}_{\nu\sigma} + \alpha\left[\widehat{{}^{(2)}T\mathring{\nabla}{}^{(3)}T} + \widehat{\mathring{R}\,{}^{(3)}T} + \dots\right], \tag{7a}$$

$${}^{(2)}S_{\mu} \approx 2\left(1 + 2{}^{(2)}\mu\right)M_{\rm Pl}{}^2\,{}^{(2)}T_{\mu} + \frac{8\alpha}{3}\mathring{\nabla}_{\nu}{}^{(2)}F_{\mu}{}^{\nu}, \tag{7b}$$

$${}^{(3)}S_{\mu} \approx -\left(1 + 8{}^{(3)}\mu\right)M_{\rm Pl}{}^2\,{}^{(3)}T_{\mu} - \frac{4\alpha}{3}\mathring{\nabla}_{\nu}{}^{(3)}F_{\mu}{}^{\nu}. \tag{7c}$$

But Eqs. (7b) and (7c) are Proca equations, whilst Eq. (7a) eliminates ${}^{(1)}\lambda_{\mu}{}^{\nu\sigma}$ in the (asymmetric) $e^{i}{}_{\mu}$-equation. The antisymmetric part is then an identity; the symmetric part is the $g_{\mu\nu}$-equation of the following effective torsion-free theory, to be compared with Eq. (6)

$$\begin{aligned} &L_{\rm PV}\left({}^{(3)}\lambda^{\mu}\to 0\right) + {}^{(3)}\mu M_{\rm Pl}{}^2\,{}^{(3)}T^{\mu}\,{}^{(3)}T_{\mu} \\ &\cong -\frac{M_{\rm Pl}{}^2\mathring{R}}{2} + \frac{2\alpha}{9}{}^{(2)}F_{\mu\nu}{}^{(2)}F^{\mu\nu} - \frac{\alpha}{2}{}^{(3)}F_{\mu\nu}{}^{(3)}F^{\mu\nu} \\ &+ \frac{M_{\rm Pl}{}^2\left(1 + 2{}^{(2)}\mu\right)}{3}{}^{(2)}T_{\mu}{}^{(2)}T^{\mu} - \frac{3M_{\rm Pl}{}^2\left(1 + 8{}^{(3)}\mu\right)}{4}{}^{(3)}T_{\mu}{}^{(3)}T^{\mu} \\ &- \frac{1}{3}{}^{(2)}T_{\mu}{}^{(2)}S^{\mu} - \frac{3}{2}{}^{(3)}T_{\mu}{}^{(3)}S^{\mu} + L_{\rm M}\left(\mathring{\Gamma}\right). \end{aligned} \tag{8}$$

In (8), we confirm for full consistency that the residual torsion reduces to the Proca pair in Eqs. (7b) and (7c), one of which is a ghost, and the full model $L_{\rm PV}$ or $L_{\rm PV}\left((2) \rightleftarrows (3)\right)$ kills off the ghost in either case. In contrast, it is critical to understand that ${}^{(1)}\lambda^{\lambda}{}_{\mu\nu}$ does not merely *kill off* the strongly coupled modes: both Eq. (6) and Eq. (8) propagate six extra non-graviton d.o.f, so ${}^{(1)}\lambda^{\lambda}{}_{\mu\nu}$ weakens the strong coupling. In Eq. (3), valid for $\alpha < 0$, the mass of ${}^{(2)}T_{\mu}$ is

$${}^{(2)}m^2 \equiv -3M_{\rm Pl}{}^2(1 + 2{}^{(2)}\mu)/4\alpha. \tag{9}$$

Perfectly analogous results hold for $\alpha > 0$, and the ${}^{(3)}T_{\mu}$ mass is ${}^{(3)}m^2 \equiv -3M_{\rm Pl}{}^2\left(1 + 8{}^{(3)}\mu\right)/4\alpha$.

V. Details of strong coupling alleviation

We briefly explain the mechanism, assuming familiarity with the Dirac algorithm [169, 191, 207–210] in which a theory has *constraints* C_i (see pedagogical introductions in [131, 211]). We target the case $L = L_{\rm PV}\left((2) \rightleftarrows (3)\right)$ corresponding to $\alpha > 0$, while referring to Fig. 2. Let $\phi^{J^P}_{X} \approx 0$ denote the *primary* constraint caused by the spin-J, parity-P part of the X-field momentum when $\pi^{J^P}_{X} \equiv \partial(eL)/\partial\dot{X}^{J^P}$ is independent of velocity $\dot{X}^{J^P}$. Introduce $u^{J^P}_{X}$ in lieu of $\dot{X}^{J^P}$, then the