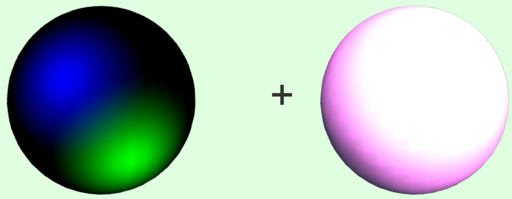
```
In[•]:= Get@FileNameJoin@{NotebookDirectory[], "MinimalMassiveOddScalar.m"};
```

Graviton plus minimal massive odd-parity scalar particle



Heavy pseudoscalar tordion   Einstein's graviton   (1)

Key observation: These calculations are designed for your project. Throughout, commentary takes the form of green text. Citations, where needed, will be managed by direct reference to arXiv numbers. One exception is the source referred to throughout as `Blagojević'; this pertains to the book `Gravitation and Gauge Symmetries'.

Loading HiGGS and GeoHiGGS

For these calculations, we will use the HiGGS and GeoHiGGS packages. Note that GeoHiGGS was not developed for public release, and so is not documented. The versions of HiGGS and GeoHiGGS used for the computations here are both developer-only, and so we include copies of the sources with the tarball.

All the requisite packages have now been loaded, so we can proceed with the computations.

1. Deriving the field equations

HiGGS is designed to study the full ten-parameter Poincaré gauge theory, including nine extra parameters which activate various Lagrange multipliers as defined in arXiv:2205.13534. As a first step, we define the most general case of the set of theories we are interested in by constructing a rule which constrains the Lagrangian couplings.

$$\Big\{\hat{\alpha}_1 \to 0,\ \hat{\alpha}_2 \to 0,\ \hat{\alpha}_4 \to 0,\ \hat{\alpha}_5 \to 0,\ \hat{\alpha}_6 \to 0,\ \mathcal{M}_{Pl}^2\,\hat{\beta}_2 \to -2\,\mathcal{M}_{Pl}^2\,\hat{\beta}_1,\ \mathcal{M}_{Pl}^2\,\hat{\beta}_3 \to -\frac{\mathcal{M}_{Pl}^2\,\hat{\beta}_1}{2},$$
$$\bar{\alpha}_1 \to 0,\ \bar{\alpha}_2 \to 0,\ \bar{\alpha}_3 \to 0,\ \bar{\alpha}_4 \to 0,\ \bar{\alpha}_5 \to 0,\ \bar{\alpha}_6 \to 0,\ \mathcal{M}_{Pl}^2\,\bar{\beta}_1 \to 0,\ \mathcal{M}_{Pl}^2\,\bar{\beta}_2 \to 0,\ \mathcal{M}_{Pl}^2\,\bar{\beta}_3 \to 0\Big\} \quad (2)$$

These rules are used to disable most of the coupling in the general theory. The couplings which are not suppressed are those which appear in the Lagrangian. Specifically the fifth alpha-hat coupling, which mediates the quadratic Riemann-Cartan-Maxwell invariant. These remaining couplings will appear in the equations below.

The generalised momenta

Having done this, we define the generalised momenta associated with this subset of multiplier-constrained Poincaré gauge theory. These quantities are defined on p. 50 of Blagojević.

The translational generalised momenta.

$$\pi_{b_i}{}^{\hbar l} \tag{3}$$

$$4\mathcal{M}_{\text{Pl}}^2 \underset{1}{\hat{\beta}} \, \delta_i^l \, \mathcal{J}\mathcal{N}\mathcal{T}^{a\hbar}{}_a - 4\mathcal{M}_{\text{Pl}}^2 \underset{1}{\hat{\beta}} \, \delta_i^\hbar \, \mathcal{J}\mathcal{N}\mathcal{T}^{al}{}_a - 2\mathcal{M}_{\text{Pl}}^2 \underset{1}{\hat{\beta}} \mathcal{J}\mathcal{N}\mathcal{T}_i{}^{\hbar l} - 2\mathcal{M}_{\text{Pl}}^2 \underset{1}{\hat{\beta}} \mathcal{J}\mathcal{N}\mathcal{T}^{\hbar}{}_i{}^{l} + 2\mathcal{M}_{\text{Pl}}^2 \underset{1}{\hat{\beta}} \mathcal{J}\mathcal{N}\mathcal{T}^{l}{}_i{}^{\hbar} \tag{4}$$

The rotational generalised momenta.

$$\pi_{\mathcal{A}_{ij}}{}^{\hbar l} \tag{5}$$

$$-\mathcal{M}_{\text{Pl}}^2 \, \delta_i^l \, \delta_j^\hbar \, \mathcal{J}\mathcal{N} + \mathcal{M}_{\text{Pl}}^2 \, \delta_i^\hbar \, \delta_j^l \, \mathcal{J}\mathcal{N} - \frac{4}{3} \underset{3}{\hat{\alpha}} \mathcal{J}\mathcal{N}\mathcal{R}_{ij}{}^{\hbar l} + \frac{4}{3} \underset{3}{\hat{\alpha}} \mathcal{J}\mathcal{N}\mathcal{R}^{\hbar}{}_i{}^{l}{}_j - \frac{4}{3} \underset{3}{\hat{\alpha}} \mathcal{J}\mathcal{N}\mathcal{R}^{l}{}_i{}^{\hbar}{}_j - \frac{4}{3} \underset{3}{\hat{\alpha}} \mathcal{J}\mathcal{N}\mathcal{R}^{\hbar}{}_j{}^{l}{}_i + \frac{4}{3} \underset{3}{\hat{\alpha}} \mathcal{J}\mathcal{N}\mathcal{R}^{l}{}_j{}^{\hbar}{}_i - \frac{4}{3} \underset{3}{\hat{\alpha}} \mathcal{J}\mathcal{N}\mathcal{R}^{\hbar l}{}_{ij} \tag{6}$$

These generalised momenta are obtained from the Lagrangian represented in Eq. (4) of arXiv:2205.13534, with most of the coupling constants set to zero as per the above restrictions. Note that this Lagrangian differs from the most general Lagrangian represented in Blagojević by the use of so-called geometric multiplier fields. These are multipliers which can disable all of the Riemann-Cartan or torsion tensors, but which may typically only be used piecemeal to disable select portions of said tensors. It is this latter use-case which we reaslise in our letter, disabling only the tensor part of the torsion.

Another note to be made here is about the caligraphic J and N symbols. Their appearance is an unfortunate side-effect of recycling the Hamiltonian-based HiGGS package to work on Lagrangian problems: these quantities are the measure on the foliation and the Lapse function, respectively, and their product is simply the determinant of the translational gauge field, (which is equivalent to the square root of the negative metric determinant).

The stress-energy tensor

Define the stress-energy tensor equation. In general, we will refer to an `equation' as an xTensor expression which we understand to vanish on the shell, accordingly we do not always make the equality explicit.

In terms of the generalised momenta, the Riemann-Cartan curvature, the torsion, and the gauge fields, and also in terms of projection operators which are used to define the various quadratic invariants, the left hand side of the stress-energy equation as expressed in the first line of Eq. (3.24b) on page 50 of Blagojević is as follows.

$$
\begin{aligned}
&-\mathcal{A}_{ijm}\,\pi_{b}{}^{j\hbar l}\,h_{\hbar}{}^{n}\,h_{l}{}^{m}-\frac{1}{2}\mathcal{M}_{Pl}{}^{2}\,\gamma^{j\hbar}\,\gamma^{lm}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,\mathcal{R}_{jl\hbar m}-\frac{1}{2}\pi_{\mathcal{A}}{}^{\hbar lm}{}_{j}\,h^{jn}\,\mathcal{R}_{\hbar lim}+\\
&\hat{\alpha}_{1}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{1}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{j\hbar lm}\,\mathcal{R}_{pqwx}+\hat{\alpha}_{2}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{2}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{j\hbar lm}\,\mathcal{R}_{pqwx}+\\
&\hat{\alpha}_{3}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{3}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{j\hbar lm}\,\mathcal{R}_{pqwx}+\hat{\alpha}_{4}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{4}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{j\hbar lm}\,\mathcal{R}_{pqwx}+\\
&\hat{\alpha}_{5}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{5}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{j\hbar lm}\,\mathcal{R}_{pqwx}+\hat{\alpha}_{6}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{6}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{j\hbar lm}\,\mathcal{R}_{pqwx}+\\
&\bar{\alpha}_{1}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{1}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{pqwx}\,\lambda_{\mathcal{R}}{}_{j\hbar lm}+\bar{\alpha}_{2}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{2}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{pqwx}\,\lambda_{\mathcal{R}}{}_{j\hbar lm}+\\
&\bar{\alpha}_{3}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{3}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{pqwx}\,\lambda_{\mathcal{R}}{}_{j\hbar lm}+\bar{\alpha}_{4}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{4}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{pqwx}\,\lambda_{\mathcal{R}}{}_{j\hbar lm}+\\
&\bar{\alpha}_{5}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{5}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{pqwx}\,\lambda_{\mathcal{R}}{}_{j\hbar lm}+\bar{\alpha}_{6}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{6}\hat{\mathcal{P}}_{\mathcal{R}}{}^{j\hbar lmpqwx}\,\mathcal{R}_{pqwx}\,\lambda_{\mathcal{R}}{}_{j\hbar lm}-\pi_{b}{}^{j\hbar l}\,h_{l}{}^{n}\,\mathcal{T}_{ji\hbar}+\\
&\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{1}\hat{\mathcal{P}}_{\mathcal{T}}{}^{j\hbar lmpq}\,\mathcal{T}_{j\hbar l}\,\mathcal{T}_{mpq}+\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{2}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{2}\hat{\mathcal{P}}_{\mathcal{T}}{}^{j\hbar lmpq}\,\mathcal{T}_{j\hbar l}\,\mathcal{T}_{mpq}+\\
&\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{3}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{3}\hat{\mathcal{P}}_{\mathcal{T}}{}^{j\hbar lmpq}\,\mathcal{T}_{j\hbar l}\,\mathcal{T}_{mpq}+\mathcal{M}_{Pl}{}^{2}\,\bar{\beta}_{1}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{1}\hat{\mathcal{P}}_{\mathcal{T}}{}^{j\hbar lmpq}\,\mathcal{T}_{mpq}\,\lambda_{\mathcal{T}}{}_{j\hbar l}+\\
&\mathcal{M}_{Pl}{}^{2}\,\bar{\beta}_{2}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{2}\hat{\mathcal{P}}_{\mathcal{T}}{}^{j\hbar lmpq}\,\mathcal{T}_{mpq}\,\lambda_{\mathcal{T}}{}_{j\hbar l}+\mathcal{M}_{Pl}{}^{2}\,\bar{\beta}_{3}\,h_{i}{}^{n}\,\mathcal{J}\mathcal{N}\,{}_{3}\hat{\mathcal{P}}_{\mathcal{T}}{}^{j\hbar lmpq}\,\mathcal{T}_{mpq}\,\lambda_{\mathcal{T}}{}_{j\hbar l}-\\
&h^{jn}\,h^{\hbar l}\,\partial_{l}\pi_{b}{}_{ij\hbar}+\pi_{b}{}_{i}{}^{j\hbar}\,h_{j}{}^{m}\,h_{\hbar}{}^{p}\,h^{ln}\,\partial_{p}b_{lm}+\pi_{b}{}_{i}{}^{j\hbar}\,h_{j}{}^{n}\,h_{\hbar}{}^{l}\,h^{mp}\,\partial_{p}b_{ml}
\end{aligned}
\tag{7}
$$

We now impose the restriction on the coupling constants to go over to the most general case studied, and then we expand the projection operators and the generalised momenta. We subtract the right hand side, i.e. the (asymmetric) matter stress-energy tensor, to form the stress-energy equation.

$$
\begin{aligned}
&-\mathcal{M}_{Pl}{}^{2}\,h^{jn}\,\mathcal{R}_{j}{}^{\hbar}{}_{i\hbar}+\frac{1}{2}\mathcal{M}_{Pl}{}^{2}\,h_{i}{}^{n}\,\mathcal{R}^{j\hbar}{}_{j\hbar}-\frac{1}{6}\hat{\alpha}_{3}\,h_{i}{}^{n}\,\mathcal{R}_{j\hbar lm}\,\mathcal{R}^{j\hbar lm}+\frac{2}{3}\hat{\alpha}_{3}\,h_{i}{}^{n}\,\mathcal{R}_{jl\hbar m}\,\mathcal{R}^{j\hbar lm}+\\
&\frac{4}{3}\hat{\alpha}_{3}\,h^{jn}\,\mathcal{R}_{j\hbar lm}\,\mathcal{R}^{\hbar l\,m}{}_{i}+\frac{2}{3}\hat{\alpha}_{3}\,h^{jn}\,\mathcal{R}_{jm\hbar l}\,\mathcal{R}^{\hbar l\,m}{}_{i}+\frac{2}{3}\hat{\alpha}_{3}\,h^{jn}\,\mathcal{R}_{\hbar ljm}\,\mathcal{R}^{\hbar l\,m}{}_{i}-\frac{4}{3}\hat{\alpha}_{3}\,h^{jn}\,\mathcal{R}_{\hbar mjl}\,\mathcal{R}^{\hbar l\,m}{}_{i}-\\
&\frac{1}{6}\hat{\alpha}_{3}\,h_{i}{}^{n}\,\mathcal{R}^{j\hbar lm}\,\mathcal{R}_{lmj\hbar}-\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h^{jn}\,\mathcal{T}_{i}{}^{\hbar l}\,\mathcal{T}_{j\hbar l}-\frac{1}{2}\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h_{i}{}^{n}\,\mathcal{T}_{j\hbar l}\,\mathcal{T}^{j\hbar l}-\\
&\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h_{i}{}^{n}\,\mathcal{T}^{j\hbar l}\,\mathcal{T}_{\hbar jl}+2\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h^{jn}\,\mathcal{T}_{\hbar jl}\,\mathcal{T}^{\hbar l}{}_{i}+2\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h^{jn}\,\mathcal{T}^{\hbar l}{}_{i}\,\mathcal{T}_{lj\hbar}-\\
&2\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h^{jn}\,\mathcal{T}_{ij}{}^{\hbar}\,\mathcal{T}^{l}{}_{\hbar l}-2\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h^{jn}\,\mathcal{T}_{ji}{}^{\hbar}\,\mathcal{T}^{l}{}_{\hbar l}+2\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h_{i}{}^{n}\,\mathcal{T}^{j}{}_{j}{}^{\hbar}\,\mathcal{T}^{l}{}_{\hbar l}+\\
&2\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h^{jn}\,\mathcal{T}^{\hbar}{}_{ij}\,\mathcal{T}^{l}{}_{\hbar l}+\frac{\tau(\Delta+\mathcal{K})^{n}{}_{i}}{\mathcal{J}\mathcal{N}}+4\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h_{i}{}^{j}\,h^{\hbar n}\left(\mathcal{D}_{j}\mathcal{T}^{l}{}_{\hbar l}\right)-4\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h_{i}{}^{n}\,h^{j\hbar}\left(\mathcal{D}_{\hbar}\mathcal{T}^{l}{}_{jl}\right)-\\
&2\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h^{jn}\,h^{\hbar l}\left(\mathcal{D}_{l}\mathcal{T}_{ij\hbar}\right)-2\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h^{jn}\,h^{\hbar l}\left(\mathcal{D}_{l}\mathcal{T}_{ji\hbar}\right)+2\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\,h^{jn}\,h^{\hbar l}\left(\mathcal{D}_{l}\mathcal{T}_{\hbar ij}\right)==0
\end{aligned}
\tag{8}
$$

This equation is nearly ready to use, but the (asymmetric) stress-energy tensor of matter first needs some attention. This is because it still depends on the contorsion, and later on in the analysis we will need to refer only to the metric-based part, from which the symmetric Einstein stress-energy tensor is eventually derived. Before moving on to the spin equations therefore, we will explore this matter.

Define the contorsion tensor to have strictly two anholonomic indices and a holonomic index, in line with its interpretation as a part of the connection. This is recovered through Eq. (3.32b) on p. 57 of Blagojević, with comparisons with Eq (3.46) on p. 61, and checks against the methods used on p. 67. Great care must be taken when attempting to achieve this index configuration using contractions of the HiGGS anholonomic torsion tensor with the translational gauge field and inverse:

$$\mathcal{K}^{ij}{}_{m} \tag{9}$$

$$\frac{1}{2} b^{a}{}_{m} \mathcal{T}_{a}{}^{ij} - \frac{1}{2} b^{a}{}_{m} \mathcal{T}^{ij}{}_{a} + \frac{1}{2} b^{a}{}_{m} \mathcal{T}^{ji}{}_{a} \tag{10}$$

Now in HiGGS we are used to using the following stress-energy tensor.

$$\tau(\Delta+\mathcal{K})^{m}{}_{\hbar} \tag{11}$$

But this tensor, the (negative) variational derivative of the matter Lagrangian (density) as defined in Eq. (3.21) on p. 48 of Blagojević, still depends on the rotational gauge field. In the second order formalism, this means that it will depend both on the Ricci rotation coefficients and on the contorsion.

We now define the part which depends only on the Ricci rotation coefficients.

To understand how the contorsion dependency enters in, we look at Eq. (3.75b) on p. 66 of Blagojević. This separated Lagrangian is varied with respect to the translational gauge field, in such a way that the Ricci rotation coefficients and contorsion are held constant (they must have two Roman and one Greek index, so that they algebraically inherit the role of the rotational gauge field). The variation of the first term will give us the above quantity, which can then be expressed in terms of the Einstein tensor using the methods of p. 67 (we do this later).

The variation of the second term rests entirely on the variation of the spin tensor. This is rather suspicious, since it means that the details of the spin tensor of matter have a say in whether we can recover the second order formalism at all. However to proceed, we look to the Dirac matter spin tensor in Example 2 at the end of p. 49 of Blagojević. We define a (Lorentz) quantity which is truly independent of the gravitational variables, being composed of Grassmann numbers and (indexed) generators of the Clifford algebra.

$$\mathcal{X}^{i}{}_{j\hbar} \tag{12}$$

Now the spin tensor is defined as follows.

$$\sigma^{i}{}_{j\hbar} \tag{13}$$

$$h^{ai}\,\mathcal{J}\,\mathcal{N}\,\mathcal{X}_{aj\hbar} \tag{14}$$

The whole of the correction to the second order matter Lagrangian density now takes the following form.

$$\frac{1}{2}\,\mathcal{K}^{ij}{}_{m}\;\sigma^{m}{}_{ij} \tag{15}$$

$$\frac{1}{2}\,\mathcal{K}^{ma}{}_{j}\;h^{ij}\,\mathcal{J}\,\mathcal{N}\,\mathcal{X}_{ima} \tag{16}$$

Now we remind ourselves about the derivatives with respect to the translational gauge field of some quantities.

The lapse function.

$$\mathcal{N} \tag{17}$$

$$h^{aa'}\;\mathcal{N}\,n_{a}\;n^{b}\;\partial_{m}b_{ba'} \tag{18}$$

The spatial measure.

$$\mathcal{J} \tag{19}$$

$$h^{aa'}\;\mathcal{J}\,\partial_{m}b_{aa'}-h^{aa'}\;\mathcal{J}\,n_{a}\;n^{b}\;\partial_{m}b_{ba'} \tag{20}$$

The inverse gauge field.

$$h_{i}{}^{n} \tag{21}$$

$$-h^{a'n}\;h_{i}{}^{a}\;\partial_{m}b_{a'a} \tag{22}$$

Now that the dependence of the contorsion correction to the second-order Lagrangian on the translational gauge field has been made clear, we can use the above derivative laws to reconstruct the variational derivative of the correction.

$$-\frac{1}{2}\,\mathcal{K}^{in}{}_{a}\;h_{\hbar}{}^{a}\;\sigma^{m}{}_{in}+\frac{1}{2}\,\mathcal{K}^{ain}\;h_{\hbar}{}^{m}\;\sigma_{nai}+\tau(\Delta)^{m}{}_{\hbar} \tag{23}$$

$$-\frac{1}{2}\,\sigma^{mai}\;\mathcal{T}_{a\hbar i}+\frac{1}{4}\,b^{ai}\;h_{\hbar}{}^{m}\;\sigma_{i}{}^{na'}\;\mathcal{T}_{ana'}-\frac{1}{4}\,\sigma^{min}\;\mathcal{T}_{\hbar in}+\frac{1}{2}\,b^{ai}\;h_{\hbar}{}^{m}\;\sigma_{i}{}^{na'}\;\mathcal{T}_{naa'}+\tau(\Delta)^{m}{}_{\hbar} \tag{24}$$

We are now ready to write a rule which converts the usual HiGGS stress-energy tensor into torsion-free and torsionful parts.

$$\tau(\Delta+\mathcal{K})^{m}{}_{\hbar} \tag{25}$$

$$\frac{1}{4} b^{aa'} h_{\hbar}{}^{m} \sigma_{a'}{}^{bb'} \mathcal{T}_{abb'} - \frac{1}{2} \sigma^{maa'} \mathcal{T}_{a\hbar a'} + \frac{1}{2} b^{aa'} h_{\hbar}{}^{m} \sigma_{a'}{}^{bb'} \mathcal{T}_{bab'} - \frac{1}{4} \sigma^{maa'} \mathcal{T}_{\hbar aa'} + \tau(\Delta)^{m}{}_{\hbar} \tag{26}$$

Now we will also write a rule to invert this.

$$\tau(\Delta)^{m}{}_{\hbar} \tag{27}$$

$$-\frac{1}{4} b^{aa'} h_{\hbar}{}^{m} \sigma_{a'}{}^{bb'} \mathcal{T}_{abb'} + \frac{1}{2} \sigma^{maa'} \mathcal{T}_{a\hbar a'} - \frac{1}{2} b^{aa'} h_{\hbar}{}^{m} \sigma_{a'}{}^{bb'} \mathcal{T}_{bab'} + \frac{1}{4} \sigma^{maa'} \mathcal{T}_{\hbar aa'} + \tau(\Delta+\mathcal{K})^{m}{}_{\hbar} \tag{28}$$

Now the second order formalism `splitting' of the stress-energy current is understood, we will `split' the stress-energy equation.

$$\begin{aligned}
&-\mathcal{M}_{Pl}{}^{2} h^{jn} \mathcal{R}_{j}{}^{\hbar}{}_{i\hbar} + \frac{1}{2} \mathcal{M}_{Pl}{}^{2} h_{i}{}^{n} \mathcal{R}^{j\hbar}{}_{j\hbar} - \frac{1}{6} \hat{\alpha}_{3} h_{i}{}^{n} \mathcal{R}_{j\hbar lm} \mathcal{R}^{j\hbar lm} + \frac{2}{3} \hat{\alpha}_{3} h_{i}{}^{n} \mathcal{R}_{jl\hbar m} \mathcal{R}^{j\hbar lm} + \\
&\quad \frac{4}{3} \hat{\alpha}_{3} h^{jn} \mathcal{R}_{j\hbar lm} \mathcal{R}^{\hbar l m}{}_{i} + \frac{2}{3} \hat{\alpha}_{3} h^{jn} \mathcal{R}_{jm\hbar l} \mathcal{R}^{\hbar l m}{}_{i} + \frac{2}{3} \hat{\alpha}_{3} h^{jn} \mathcal{R}_{\hbar ljm} \mathcal{R}^{\hbar l m}{}_{i} - \frac{4}{3} \hat{\alpha}_{3} h^{jn} \mathcal{R}_{\hbar mjl} \mathcal{R}^{\hbar l m}{}_{i} - \\
&\quad \frac{1}{6} \hat{\alpha}_{3} h_{i}{}^{n} \mathcal{R}^{j\hbar lm} \mathcal{R}_{lmj\hbar} - \frac{\sigma^{nj\hbar} \mathcal{T}_{ij\hbar}}{4 \mathcal{J} \mathcal{N}} - \frac{\sigma^{nj\hbar} \mathcal{T}_{ji\hbar}}{2 \mathcal{J} \mathcal{N}} - \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h^{jn} \mathcal{T}^{\hbar l}{}_{i} \mathcal{T}_{j\hbar l} + \\
&\quad \frac{b^{j\hbar} h_{i}{}^{n} \sigma_{\hbar}{}^{lm} \mathcal{T}_{jlm}}{4 \mathcal{J} \mathcal{N}} - \frac{1}{2} \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h_{i}{}^{n} \mathcal{T}_{j\hbar l} \mathcal{T}^{j\hbar l} - \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h_{i}{}^{n} \mathcal{T}^{j\hbar l} \mathcal{T}_{\hbar jl} + \\
&\quad 2 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h^{jn} \mathcal{T}_{\hbar jl} \mathcal{T}^{\hbar l}{}_{i} + 2 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h^{jn} \mathcal{T}^{\hbar l}{}_{i} \mathcal{T}_{lj\hbar} + \frac{b^{j\hbar} h_{i}{}^{n} \sigma_{\hbar}{}^{lm} \mathcal{T}_{ljm}}{2 \mathcal{J} \mathcal{N}} - \\
&\quad 2 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h^{jn} \mathcal{T}_{ij}{}^{\hbar} \mathcal{T}^{l}{}_{\hbar l} - 2 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h^{jn} \mathcal{T}_{ji}{}^{\hbar} \mathcal{T}^{l}{}_{\hbar l} + 2 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h_{i}{}^{n} \mathcal{T}^{j}{}_{j}{}^{\hbar} \mathcal{T}^{l}{}_{\hbar l} + \\
&\quad 2 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h^{jn} \mathcal{T}^{\hbar}{}_{ij} \mathcal{T}^{l}{}_{\hbar l} + \frac{\tau(\Delta)^{n}{}_{i}}{\mathcal{J} \mathcal{N}} + 4 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h_{i}{}^{j} h^{\hbar n} \left(\mathcal{D}_{j} \mathcal{T}^{l}{}_{\hbar l}\right) - 4 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h_{i}{}^{n} h^{j\hbar} \left(\mathcal{D}_{\hbar} \mathcal{T}^{l}{}_{jl}\right) - \\
&\quad 2 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h^{jn} h^{\hbar l} \left(\mathcal{D}_{l} \mathcal{T}_{ij\hbar}\right) - 2 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h^{jn} h^{\hbar l} \left(\mathcal{D}_{l} \mathcal{T}_{ji\hbar}\right) + 2 \mathcal{M}_{Pl}{}^{2} \hat{\beta}_{1} h^{jn} h^{\hbar l} \left(\mathcal{D}_{l} \mathcal{T}_{\hbar ij}\right) == 0
\end{aligned} \tag{29}$$

This then is the final form of the stress-energy equation in the first-order formalism. How does it differ from the first version of the equation which we wrote down above? The matter stress-energy tensor has been decomposed into a metric-based part and a series of terms bilinear in the torsion and the matter spin tensor. The gravitational part of the field equations is unaffected.

Key observation: This equation is the stress-energy equation which should give rise to some of the cosmological perturbation equations. Remember that we can neglect the spin tensor, that the field strength tensors have only Roman indices, and that Greek indices are provided by the matter stress-energy tensor and the (inverse) translational gauge field wherever it appears. The factor of caligraphic J multiplied by caligraphic N is intended to mean the determinant of the translational gauge field, I've just written it out in 3+1 notation. I'd recommend contracting this equation with the tetrad to produce two raised Greek indices, and then extracting the symmetric part (the Einstein equations) and the antisymmetric part (which you should prove is merely an identity, with the help of the spin equation later on).

We also construct a replacement rule to convert the torsionless stress-energy tensor into gravitational variables, i.e. torsion, Riemann-Cartan curvature and multiplier fields, according to this stress-energy equation.

The spin tensor

Having studied the general stress-energy equation, we turn to the spin-torsion equation. The spin tensor is defined in Eq. (3.21) on p. 48 of Blagojević.

Once again, in terms of the generalised momenta and the gauge fields, the left hand side of the spin equation as expressed in the second line of Eq. (3.24b) on page 50 of Blagojević is as follows.

$$\pi_{b_{ij}}{}^{l}\, h_{l}{}^{n} - \pi_{b_{ji}}{}^{l}\, h_{l}{}^{n} - \mathcal{A}^{y}{}_{jx}\, \pi_{\mathcal{A}_{iylm}}\, h^{ln}\, h^{mx} + \mathcal{A}^{y}{}_{ix}\, \pi_{\mathcal{A}_{jylm}}\, h^{ln}\, h^{mx} + \pi_{\mathcal{A}_{ijmy}}\, h^{ln}\, h^{mx}\, h^{ya}\, \partial_{a} b_{lx} - h^{ln}\, h^{mx}\, \partial_{x} \pi_{\mathcal{A}_{ijlm}} + \pi_{\mathcal{A}_{ijly}}\, h^{ln}\, h^{mx}\, h^{ya}\, \partial_{x} b_{ma} \qquad (30)$$

We now impose the restriction on the coupling constants to go over to the most general case studied, and then we expand the projection operators and the generalised momenta. We subtract the right hand side, i.e. the matter spin tensor, to form the spin equation.

$$\begin{aligned}
&\frac{\sigma^{n}{}_{ij}}{\mathcal{J}\mathcal{N}} - 4\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\, h^{an}\, \mathcal{T}_{aij} + \mathcal{M}_{Pl}{}^{2}\, h^{an}\, \mathcal{T}_{aij} - \frac{2}{3}\hat{\alpha}_{3}\, h^{an}\, \mathcal{R}_{ijlm}\, \mathcal{T}_{a}{}^{lm} + \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, \mathcal{R}_{iljm}\, \mathcal{T}_{a}{}^{lm} - \\
&\quad \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, \mathcal{R}_{jlim}\, \mathcal{T}_{a}{}^{lm} - \frac{2}{3}\hat{\alpha}_{3}\, h^{an}\, \mathcal{R}_{lmij}\, \mathcal{T}_{a}{}^{lm} + 4\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\, h_{j}{}^{n}\, \mathcal{T}^{a}{}_{ia} - \mathcal{M}_{Pl}{}^{2}\, h_{j}{}^{n}\, \mathcal{T}^{a}{}_{ia} - \\
&\quad 4\,\mathcal{M}_{Pl}{}^{2}\,\hat{\beta}_{1}\, h_{i}{}^{n}\, \mathcal{T}^{a}{}_{ja} + \mathcal{M}_{Pl}{}^{2}\, h_{i}{}^{n}\, \mathcal{T}^{a}{}_{ja} + \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, \mathcal{R}_{amij}\, \mathcal{T}^{lm}{}_{l} - \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, \mathcal{R}_{iajm}\, \mathcal{T}^{lm}{}_{l} + \\
&\quad \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, \mathcal{R}_{ijam}\, \mathcal{T}^{lm}{}_{l} + \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, \mathcal{R}_{imja}\, \mathcal{T}^{lm}{}_{l} + \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, \mathcal{R}_{jaim}\, \mathcal{T}^{lm}{}_{l} - \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, \mathcal{R}_{jmia}\, \mathcal{T}^{lm}{}_{l} - \\
&\quad \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, h^{lm}\left(\mathcal{D}_{m}\mathcal{R}_{alij}\right) + \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, h^{lm}\left(\mathcal{D}_{m}\mathcal{R}_{iajl}\right) - \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, h^{lm}\left(\mathcal{D}_{m}\mathcal{R}_{ijal}\right) - \\
&\quad \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, h^{lm}\left(\mathcal{D}_{m}\mathcal{R}_{ilja}\right) - \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, h^{lm}\left(\mathcal{D}_{m}\mathcal{R}_{jail}\right) + \frac{4}{3}\hat{\alpha}_{3}\, h^{an}\, h^{lm}\left(\mathcal{D}_{m}\mathcal{R}_{jlia}\right) == 0
\end{aligned} \qquad (31)$$

This time we do not need to further decompose the matter sources, and so this constitutes the final form of the spin equations in the first-order formalism.

Key observation: This is the spin equation, from which additional cosmological perturbation equations can be derived. You recall that the spin equation is obtained by taking variational derivatives of the action with respect to the rotational gauge field. Hence, the indices of the equation are precisely those of the rotational gauge field itself. That means that you can go ahead and decompose this equation just as you did the rotational gauge field, and the torsion tensor example I gave you earlier. Remember that the spin tensor itself can be set to zero.

Stress-energy conservation

We now have both sets of field equations at our disposal, with reference to their respective source currents. It would be helpful at this stage to verify the second conservation law from Eq. (3.23) p. 49 in Blagojević.

$$-\frac{1}{2} b^{l}{}_{m}\, b^{q}{}_{n}\, \mathcal{R}^{ij}{}_{lq}\, \sigma^{n}{}_{ij} - b^{l}{}_{m}\, b^{q}{}_{n}\, \mathcal{T}^{\hbar}{}_{lq}\, \tau(\Delta+\mathcal{K})^{n}{}_{\hbar} - \mathcal{A}^{l}{}_{\hbar n}\, b^{\hbar}{}_{m}\, \tau(\Delta+\mathcal{K})^{n}{}_{l} + b^{\hbar}{}_{m}\, \partial_{n}\tau(\Delta+\mathcal{K})^{n}{}_{\hbar} \quad (32)$$

Substitution of the field equations into this expression, which refers to the sources only, causes it to vanish. The calculation is computationally quite expensive, so we will omit it here.

Spin conservation

Equally, it is important to verify the second conservation law from Eq. (3.23) page 49 in Blagojević.

First term in the putatively vanishing expression.

$$-b_{jn}\, \tau(\Delta)^{n}{}_{i} + b_{in}\, \tau(\Delta)^{n}{}_{j} \quad (33)$$

Second term in the putatively vanishing expression.

$$-\mathcal{A}^{l}{}_{jn}\, \sigma^{n}{}_{il} - \mathcal{A}^{l}{}_{in}\, \sigma^{n}{}_{lj} + \partial_{n}\sigma^{n}{}_{ij} \quad (34)$$

Once again, by substituting for the source currents on the field equation shell, we expect these terms to cancel, but the calculation itself is omitted for brevity.

Irreducible decomposition of spin equations

Assuming that the field equations have thus been correctly obtained, we find it important to decompose them into their irreducible parts under the actions of the Lorentz group.

The largest and most cumbersome part of the spin equation, the tensor part with 16 degrees of freedom.

$$-\frac{b_{w}{}^{a}\, \sigma^{r}{}_{au}}{3\mathcal{T}\mathcal{N}} + \frac{b_{u}{}^{a}\, \sigma^{r}{}_{aw}}{3\mathcal{T}\mathcal{N}} + \frac{2\, b^{ra}\, \sigma_{auw}}{3\mathcal{T}\mathcal{N}} - \frac{b^{ai}\, \delta^{r}{}_{w}\, \sigma_{iua}}{3\mathcal{T}\mathcal{N}} + \frac{b^{ai}\, \delta^{r}{}_{u}\, \sigma_{iwa}}{3\mathcal{T}\mathcal{N}} +$$
$$\frac{4}{9}\hat{\alpha}_{3}\, \delta^{r}{}_{w}\, \mathcal{R}_{aiuj}\, \mathcal{T}^{aij} - \frac{4}{9}\hat{\alpha}_{3}\, \delta^{r}{}_{u}\, \mathcal{R}_{aiwj}\, \mathcal{T}^{aij} + \frac{2}{9}\hat{\alpha}_{3}\, \delta^{r}{}_{w}\, \mathcal{R}_{ijua}\, \mathcal{T}^{aij} - \frac{2}{9}\hat{\alpha}_{3}\, \delta^{r}{}_{u}\, \mathcal{R}_{ijwa}\, \mathcal{T}^{aij} +$$
$$\frac{2}{9}\hat{\alpha}_{3}\, \delta^{r}{}_{w}\, \mathcal{R}_{uaij}\, \mathcal{T}^{aij} - \frac{4}{9}\hat{\alpha}_{3}\, \delta^{r}{}_{w}\, \mathcal{R}_{uiaj}\, \mathcal{T}^{aij} - \frac{2}{9}\hat{\alpha}_{3}\, \delta^{r}{}_{u}\, \mathcal{R}_{waij}\, \mathcal{T}^{aij} + \frac{4}{9}\hat{\alpha}_{3}\, \delta^{r}{}_{u}\, \mathcal{R}_{wiaj}\, \mathcal{T}^{aij} +$$

$$\frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_1\,\delta^r{}_w\,\mathcal{T}^a{}_{ua} - \frac{1}{3}\mathcal{M}_{Pl}^2\,\delta^r{}_w\,\mathcal{T}^a{}_{ua} - \frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_1\,\delta^r{}_u\,\mathcal{T}^a{}_{wa} + \frac{1}{3}\mathcal{M}_{Pl}^2\,\delta^r{}_u\,\mathcal{T}^a{}_{wa} -$$
$$\frac{4}{9}\hat{\alpha}_3\,\mathcal{R}_{aiuw}\,\mathcal{T}^{rai} + \frac{8}{9}\hat{\alpha}_3\,\mathcal{R}_{uawi}\,\mathcal{T}^{rai} - \frac{4}{9}\hat{\alpha}_3\,\mathcal{R}_{uwai}\,\mathcal{T}^{rai} - \frac{8}{9}\hat{\alpha}_3\,\mathcal{R}_{waui}\,\mathcal{T}^{rai} -$$
$$\frac{8}{3}\mathcal{M}_{Pl}^2\hat{\beta}_1\,\mathcal{T}^r{}_{uw} + \frac{2}{3}\mathcal{M}_{Pl}^2\,\mathcal{T}^r{}_{uw} - \frac{2}{9}\hat{\alpha}_3\,\mathcal{R}_{ai}{}^r{}_w\,\mathcal{T}_u{}^{ai} + \frac{4}{9}\hat{\alpha}_3\,\mathcal{R}^r{}_{awi}\,\mathcal{T}_u{}^{ai} - \frac{2}{9}\hat{\alpha}_3\,\mathcal{R}^r{}_{wai}\,\mathcal{T}_u{}^{ai} -$$
$$\frac{4}{9}\hat{\alpha}_3\,\mathcal{R}_{wa}{}^r{}_i\,\mathcal{T}_u{}^{ai} - \frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_1\,\mathcal{T}_u{}^r{}_w + \frac{1}{3}\mathcal{M}_{Pl}^2\,\mathcal{T}_u{}^r{}_w + \frac{2}{9}\hat{\alpha}_3\,\mathcal{R}_{ai}{}^r{}_u\,\mathcal{T}_w{}^{ai} - \frac{4}{9}\hat{\alpha}_3\,\mathcal{R}^r{}_{aui}\,\mathcal{T}_w{}^{ai} +$$
$$\frac{2}{9}\hat{\alpha}_3\,\mathcal{R}^r{}_{uai}\,\mathcal{T}_w{}^{ai} + \frac{4}{9}\hat{\alpha}_3\,\mathcal{R}_{ua}{}^r{}_i\,\mathcal{T}_w{}^{ai} + \frac{4}{3}\mathcal{M}_{Pl}^2\hat{\beta}_1\,\mathcal{T}_w{}^r{}_u - \frac{1}{3}\mathcal{M}_{Pl}^2\,\mathcal{T}_w{}^r{}_u == 0$$

The simplest vector part of the spin equation, with four degrees of freedom.

$$-\frac{b^{al}\,\sigma_{lwa}}{\mathcal{J}\,\mathcal{N}} + \frac{4}{3}\hat{\alpha}_3\,\mathcal{R}_{alwm}\,\mathcal{T}^{alm} + \frac{2}{3}\hat{\alpha}_3\,\mathcal{R}_{lmwa}\,\mathcal{T}^{alm} +$$
$$\frac{2}{3}\hat{\alpha}_3\,\mathcal{R}_{walm}\,\mathcal{T}^{alm} - \frac{4}{3}\hat{\alpha}_3\,\mathcal{R}_{wlam}\,\mathcal{T}^{alm} - 8\,\mathcal{M}_{Pl}^2\hat{\beta}_1\,\mathcal{T}^a{}_{wa} + 2\,\mathcal{M}_{Pl}^2\,\mathcal{T}^a{}_{wa} == 0 \quad (36)$$

The other simplest axial vector part of the spin equation, also with four degrees of freedom.

$$-\frac{b^{ai}\,\epsilon\gamma^c{}_{aru}\,\sigma_i{}^{ru}}{\mathcal{J}\,\mathcal{N}} + 4\,\hat{\alpha}_3\,\epsilon\gamma^c{}_{ruw}\,\mathcal{R}_i{}^{ruw}\,\mathcal{T}^a{}_a{}^i + 4\,\hat{\alpha}_3\,\epsilon\gamma^c{}_{ruw}\,\mathcal{R}^{ru}{}_i{}^w\,\mathcal{T}^a{}_a{}^i + 4\,\mathcal{M}_{Pl}^2\hat{\beta}_1\,\epsilon\gamma^c{}_{air}\,\mathcal{T}^{air} -$$
$$\mathcal{M}_{Pl}^2\,\epsilon\gamma^c{}_{air}\,\mathcal{T}^{air} + \frac{2}{3}\hat{\alpha}_3\,\epsilon\gamma^c{}_{auw}\,\mathcal{R}_{ir}{}^{uw}\,\mathcal{T}^{air} - \frac{8}{3}\hat{\alpha}_3\,\epsilon\gamma^c{}_{auw}\,\mathcal{R}_i{}^u{}_r{}^w\,\mathcal{T}^{air} +$$
$$\frac{2}{3}\hat{\alpha}_3\,\epsilon\gamma^c{}_{auw}\,\mathcal{R}^{uw}{}_{ir}\,\mathcal{T}^{air} - 4\,\hat{\alpha}_3\,\epsilon\gamma^c{}_{ruw}\,h^{ai}\left(\mathcal{D}_i\mathcal{R}_a{}^{ruw}\right) - 4\,\hat{\alpha}_3\,\epsilon\gamma^c{}_{ruw}\,h^{ai}\left(\mathcal{D}_i\mathcal{R}^{ru}{}_a{}^w\right) == 0 \quad (37)$$

Here is a quick reminder of our irrep conventions for the torsion.

$$\frac{2}{3}\,{}^{1}\mathcal{T}^m{}_{ns} - \frac{2}{3}\,{}^{1}\mathcal{T}^m{}_{sn} - \frac{1}{3}\,\delta^m{}_s\,{}^{2}\mathcal{T}_n + \frac{1}{3}\,\delta^m{}_n\,{}^{2}\mathcal{T}_s + \epsilon\gamma^m{}_{nsa}\,{}^{3}\mathcal{T}^a \quad (38)$$

And we also want to see what the teleparallel Lagrangian looks like, this is the double-bar T symbol which must be multiplied by the measure and half the Planck mass, with a positive sign (see eq. (15) in arXiv:2006.03581).

$$\frac{4}{9}\,{}^{1}\mathcal{T}_{ijk}\,{}^{1}\mathcal{T}^{ijk} - \frac{4}{9}\,{}^{1}\mathcal{T}_{ikj}\,{}^{1}\mathcal{T}^{ijk} - \frac{2}{3}\,{}^{2}\mathcal{T}_i\,{}^{2}\mathcal{T}^i + \frac{3}{2}\,{}^{3}\mathcal{T}_i\,{}^{3}\mathcal{T}^i \quad (39)$$

We will also take a look at our irrep conventions for the spin tensor.

$$h^k{}_m\,\sigma^m{}_{ij} \quad (40)$$

$$\frac{2}{3}\,{}^{1}\!\sigma_{i\ \ j}^{\ ħ} - \frac{2}{3}\,{}^{1}\!\sigma_{j\ \ i}^{\ ħ} - \frac{1}{3}\,\delta_{j}^{ħ}\,{}^{2}\!\sigma_{i} + \frac{1}{3}\,\delta_{i}^{ħ}\,{}^{2}\!\sigma_{j} - \epsilon\gamma_{ija}^{\ \ \ ħ}\,{}^{3}\!\sigma^{a} \tag{41}$$

Some care has to be taken when understanding how the vector and axial vector parts of the torsion couple to the respective parts of the spin tensor. The reason for this is that the spin tensor is constructed, as per the discussion above, with two Lorentz indices, and so in the stress-energy field equation there is a possible factor of two that can go missing unless the indices are kept carefully tracked. When we go over to the GeoHiGGS second-order formulation in the next part of the script, all appearances of the translational gauge field and its inverse are simply replaced by the Kroneker symbol. This is safe in the gravity sector of the theory, but not in the matter coupling.

Now we assume that every instance of an epsilon tensor appears with lowered Greek indices in the second-order formalism, so that it is safe to consider this equal to the original epsilon (with Roman indices assumed) with all indices contracted with translational (not inverse) gauge fields.

{HoldPattern[$\epsilon\gamma^{\underline{\underline{abcd}}}$] :→ Module[{xAct`HiGGS`Private`Zz9$15458, xAct`HiGGS`Private`Zz9$15459, xAct`HiGGS`Private`Zz9$15460, xAct`HiGGS`Private`Zz9$15461}, $b^{\text{xAct`HiGGS`Private`Zz9\$15458}a}$ $b^{\text{xAct`HiGGS`Private`Zz9\$15459}b}$ $b^{\text{xAct`HiGGS`Private`Zz9\$15460}c}$ $b^{\text{xAct`HiGGS`Private`Zz9\$15461}d}$ $\epsilon\gamma_{\text{xAct`HiGGS`Private`Zz9\$15458xAct`HiGGS`Private`Zz9}}$ (42)

Note that the above output is a rule to be used internally by the script, not a mathematical expression. We will occasionally display such rules below as we define them.

Moreover, Greek-index curved-metric tensors will always appear so as to satisfy the following rules, where we use the flat-space metric of HiGGS as a temporary abuse of notation.

{HoldPattern[$b^{\underline{\underline{i}}\underline{\underline{m}}}\ b^{\underline{\underline{j}}}_{\ \underline{\underline{m}}}$] :→ Module[{}, γ^{ij}], HoldPattern[$b^{\underline{\underline{i}}}_{\ \underline{\underline{m}}}\ b^{\underline{\underline{j}}\underline{\underline{m}}}$] :→ Module[{}, γ^{ij}]} (43)

{HoldPattern[$h^{\underline{\underline{i}}\underline{\underline{m}}}\ h^{\underline{\underline{j}}}_{\ \underline{\underline{m}}}$] :→ Module[{}, γ^{ji}], HoldPattern[$h^{\underline{\underline{i}}}_{\ \underline{\underline{m}}}\ h^{\underline{\underline{j}}\underline{\underline{m}}}$] :→ Module[{}, γ^{ji}]} (44)

Here is the part of the Lagrangian in which the torsion is coupled to spin.

$$\frac{1}{4}\,h^{m}_{\ \ z}\,\sigma^{z}_{\ \ ij}\left(b^{i}_{\ \mathfrak{s}}\,h^{je}\,h_{m}^{\ \ m'}\,\mathcal{T}^{\mathfrak{s}}_{\ \ em'} + b^{j}_{\ \mathfrak{s}}\,h^{ie}\,h_{m}^{\ \ m'}\,\mathcal{T}^{\mathfrak{s}}_{\ \ m'e} - b_{m\mathfrak{s}}\,h^{ir}\,h^{jp}\,\mathcal{T}^{\mathfrak{s}}_{\ \ rp}\right) \tag{45}$$

Here is the expression expanded according to the above rules.

$$-\frac{4}{9}\,b^{ei}\,h^{jm}\,h^{m'p}\,{}^{1}\!\sigma_{ejm'}\,{}^{1}\!\mathcal{T}_{imp} + \frac{1}{3}\,b^{ei}\,\epsilon\gamma_{ejm'r}\,h^{jm}\,h^{m'p}\,{}^{3}\!\sigma^{r}\,{}^{1}\!\mathcal{T}_{imp} + \frac{4}{9}\,b^{ei}\,h^{jm}\,h^{m'p}\,{}^{1}\!\sigma_{ejm'}\,{}^{1}\!\mathcal{T}_{ipm} - \frac{1}{3}\,h^{ei}\,{}^{2}\!\sigma_{e}\,{}^{2}\!\mathcal{T}_{i} - \frac{3}{2}\,b^{ei}\,{}^{3}\!\sigma_{e}\,{}^{3}\!\mathcal{T}_{i} \tag{46}$$

The factors of the translational gauge field which appear here will be needed when we reconstruct the effective second-order theory at the end of the script, but this process will not be made explicit. Moreover, we note that it is better to assume a very general form for the matter spin tensor, such that the specific couplings between different fermion currents and the torsion are basically obscured.