Gauge Theories of Gravity

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This work is dedicated to the memory of Donald Lynden-Bell.

Gravitation – M. C. Escher, 1952

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This thesis contains fewer than 75,000 words including appendices, bibliography, footnotes, tables, equations and an extension of 15,000 words generously granted prior to submission by the Degree Committee for the Faculty of Physics & Chemistry. Due to length limitations, the content of this thesis is not exhaustively representative of the research conducted during the funded period.

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Abstract

A novel alternative to Einstein's general theory of relativity is presented, based on the gauge principle. The gravitational Lagrangian has no Einstein–Hilbert term, but is constructed from quadratic invariants of the Riemann–Cartan curvature and torsion, constituting a geometric gauge theory of the Poincaré group. Despite the introduction of Planck and cosmological constant scales through the torsion couplings, the linearised free theory is not only unitary but also power-counting renormalisable. A conformal symmetry in this regime is broken naturally in the nonlinear cosmology, for which constant axial torsion is an attractor state of the background. The Hubble dynamics are then identical to those of Friedmann up to a complete screening of the spatial curvature, and a boundary condition which can dynamically replicate the effects of dark radiation in the early Universe. In a simplified version of the theory, part of the torsion is removed via multipliers without detriment to the known phenomenology. This procedure introduces classical ghosts to the Minkowski background, but produces the Newtonian limit on the axial vector background anticipated in Nature.

Chapter 1 By way of an introduction, Einstein's theory is first explored via gravitational energy localisation. It is shown that Butcher's recent localisation scheme for weak gravity generalises to Einstein's pseudotensor. By minimally extending general relativity to the Einstein–Cartan Poincaré gauge theory, the energy of a static, spherical spacetime is shown to adopt a Klein–Gordon form.

Chapter 2 The cosmologies of general nonminimal Poincaré and extended Weyl gauge theories are charted. A (weak-field) unitary and renormalisable special case is shown to produce Einsteinian cosmology and dynamically emergent dark radiation from a purely quadratic Lagrangian. Dark radiation has been of some interest as a candidate, partial solution to any current Hubble tension.

Chapter 3 A torsion-free scalar-tensor analogue of general Poincaré gauge theory is developed, revealing the torsionful theory to contain a non-canonical *Cuscuton* field. Using this analogue, our new theory is shown to predict dynamically emergent dark energy.

Chapter 4 A canonical analysis indicates potential nonlinear pathologies within those purely quadratic Poincaré gauge theories whose quantum mechanics appear viable at linear order. Potential problems include the activation of ghost modes and bifurcation of constraints.

Chapter 5 Multipliers are introduced to improve the canonical structure of general Poincaré gauge theory. Application to the new theory reveals a weak-field correspondence with conformal gravity. The rôle of cosmic torsion in breaking the conformal symmetry and setting the Newtonian limit is explored.

Conclusions and technical appendices follow.

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Nomenclature

(L)CDM	(Lambda-) cold dark matter
(B)SM	(beyond) standard model
(F/S)C	(first/second)-class
(J/E)F	(Jordan/Einstein) frame
(P/S/T)iC	(primary/secondary/tertiary) if-constraint
(P/W/eW/MA)GT$^{(q)(+)}$	(Poincaré/Weyl/extended Weyl/metric-affine) gauge theory built from maximally (quadratic) invariants of (positive) parity
(Q/C)FT	(quantum/conformal) field theory
ADM	Arnowitt–Deser–Misner
BAO	baryon acoustic oscillation
BBN	Big Bang nucleosynthesis
BHN	Baekler, Hehl and Nester
CG	conformal gravity
CGS	centimetre-gram-second
CMB	cosmic microwave background
CS	correspondence solution
D.o.F	degrees of freedom
E.o.(S/M)	equation of (state/motion)
ECT	Einstein–Cartan theory
EFT	effective field theory
FLRW	Friedmann–Lemaître–Robertson–Walker
GCT	general coordinate transformations
GR	general relativity

GTG	gauge theory gravity
GWS	Glashow–Weinberg–Salam
i(P/S/T)(F/S)C	(P/S/T)iC which is (F/S)C on the final shell
IR	infrared
LSZ	Lehmann–Symanzik–Zimmermann
MA	metrical analogue
MCMC	Markov chain Monte Carlo
NSI	normally scale-invariant
PCR	power-counting renormalisable
PPM	primary Poisson matrix
PPN	parametrised post-Newtonian
Q(E/C)D	quantum (electro/chromo)dynamics
RST	Riemann-squared theory
s(P/S)FC	sure (primary/secondary) constraint, which is always FC
SCP	strong cosmological principle
SET	stress-energy tensor
SNY	Shie, Nester and Yo
SPO	spin-projection operator
STA	spacetime algebra
TRGB	tip of the red giant branch
TT	transverse-traceless
UV	ultraviolet
VEV	vacuum expectation value
ZX	Zhang and Xu

Conventions

Units

We work by default in the geometrised system of reduced Planck units, in which the speed of light in a vacuum c, the gravitational constant G, the reduced Planck constant $\hbar$ and the Boltzmann constant k_{B} all have a numerical value of unity. The Einstein constant κ and reduced Planck mass m_{p} appear as derived quantities in this system

$$\kappa \equiv \frac{8\pi G}{c^4}, \quad m_{\text{p}} \equiv \sqrt{\frac{\hbar c}{8\pi G}},$$

though in practice we will mostly leave them in symbolic form to denote dimensionality wherever it arises, and use them interchangeably according to $\kappa \equiv {m_{\text{p}}}^{-2}$. For an anthropocentric perspective, we will occasionally resort to Gaussian centimetre-gram-second (CGS) units, and other units common to cosmology and particle physics.

Spacetime

We use the 'West Coast' signature $(+,-,-,-)$, with one brief exception in Appendix C.2. Our conventions for indices within various interpretations of $d = 4$ spacetime are listed in Table 1. The geometric flavours of spacetime are denoted as: Minkowski M_4, Riemann V_4, teleparallel T_4, Riemann–Cartan U_4, Weyl–Cartan Y_4 and linearly connected L_4. The manifolds $\check{\mathcal{M}}$ and $\mathcal{M}$ are generic instances of M_4 and V_4 respectively, and the vector space $\{x\}$ realises M_4 in the spacetime algebra (STA).

Table 1 Index conventions. The tensor slot notation is confined to Chapter 1.

	Dimensions		Variable rank	ADM projected
	0–3	1–3		
Penrose's holonomic slots	$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\ldots$			
Holonomic coordinate indices	$\mu, \nu, \xi\ldots$	$\alpha, \beta, \gamma\ldots$	$\acute{\mu}, \acute{\nu}, \acute{\xi}\ldots\acute{\alpha}, \acute{\beta}, \acute{\gamma}\ldots$	
Anholonomic Lorentz indices	$i, j, k\ldots$	$a, b, c\ldots$	$\acute{i}, \acute{j}, \acute{k}\ldots\acute{a}, \acute{b}, \acute{c}\ldots$	$\bar{i}, \bar{j}, \bar{k}\ldots\bar{a}, \bar{b}, \bar{c}\ldots$

Local structure

On (e.g.) $\mathcal{M}$ the chart $\{x^\mu\}$ generates the vectors $\boldsymbol{e}_\mu \equiv \partial/\partial x^\mu$, metric components $g_{\mu\nu} \equiv \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu$, covector one-forms $\boldsymbol{e}^\mu \equiv dx^\mu$ and natural volume four-form $\boldsymbol{\epsilon} \equiv \sqrt{-g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, where $g \equiv \det g_{\mu\nu}$, whose components are $\epsilon_{\mu\nu\sigma\lambda} \equiv \sqrt{-g}\varepsilon_{\mu\nu\sigma\lambda}$ where the Levi–Civita symbol is $\varepsilon^{0123} \equiv 1$. The symbol $\mathrm{d}^4x \equiv \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$ denotes a product of differentials, distinct from forms. The (overlapping) notation for the local structure of the STA is developed in Appendices A.1 and A.3.

Introduction

In light of both theoretical minimalism and empirical verification, the preferred effective theory of gravitational interaction is the general relativity (GR) of Einstein [1, 2] and Hilbert [3]

$$L_{\mathrm{T}} = -\frac{1}{2\kappa}R - \frac{\Lambda}{\kappa} + L_{\mathrm{M}}. \tag{1}$$

The gravitational portion $L_{\mathrm{G}} \equiv L_{\mathrm{T}} - L_{\mathrm{M}}$ of the total Lagrangian L_{T} is powered by the scalar part of the Riemann curvature tensor $R \equiv R^{\mu\nu}{}_{\mu\nu}$, which is the de facto gravitational field strength and second-order concomitant of the metric gravitational potential $g_{\mu\nu}$, being of the form $R \sim \partial^2 g + (\partial g)^2$. Sufficiently far below the electroweak transition temperature at $T_{\mathrm{c}} \gtrsim 1 \times 10^{11}\,\mathrm{eV}$ [4–8] [1] the matter Lagrangian L_{M} can provide the standard model (SM) of particle physics as an effective $\mathrm{SU}(3)_{\mathrm{c}} \times \mathrm{U}(1)_{\mathrm{em}}$ gauge theory, whose bosons are minimally coupled to $g_{\mu\nu}$. Metrical coupling to fermions is not defined, motivating the *tetrad* extension of GR [10], while an effective nonminimal coupling to the Higgs doublet is often assumed in the fully symmetric $\mathrm{SU}(3)_{\mathrm{c}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ theory [11–13] – and whatever may lie beyond it. As a generally covariant, second-order theory, GR is restricted by Lovelock's theorem [14] to express *two* physical scales, identified in (1) with the Einstein constant κ and the cosmological constant Λ. By supporting the general covariance property, GR also tacitly gauges the diffeomorphism group $\mathbb{R}^{1,3}$ on a curved, Riemannian spacetime [15].

Myriad astrophysical phenomena conform to the predictions of GR, ranging from the perihelion precession of Mercury [1] and solar deflection of starlight [16], to gravitational waves [17] and black holes [18]. Phenomena at astrophysical scales $\lesssim 1 \times 10^2\,\mathrm{Mpc}$ [19, 20] are chiefly sensitive to the feeble gravitational coupling κ, first measured by Cavendish in the guise of Newton's constant G [21]. In the microscopic context of a prospective quantum theory this coupling translates to the Planck mass which, at $m_{\mathrm{p}} \equiv 1/\sqrt{\kappa} \sim 1 \times 10^{-5}\,\mathrm{g}$, is so heavy that it might seem to derive from our own macroscopic world, being comparable to the weight of a mosquito egg. More formally, $m_{\mathrm{p}} = (2.435\,320 \pm 0.000\,028) \times 10^{27}\,\mathrm{eV}$ is the highest energy scale in fundamental physics [22], and is typically viewed as the relic (or gatekeeper) of an unknown quantum theory of gravity which integrates out to give the effective theory of GR.

GR is also supported on cosmological scales, and is an assumption of the prevailing *concordance* model of cosmology [23–26]. Despite clear phenomenological successes, certain theoretical aspects of this model – better known as LCDM – remain mysterious. The eponymous constant Λ is thought to dynamically dominate the current Universe as a *dark energy* density, which accelerates the Hubble flow [27–29]. Accordingly, current combined observations of the cosmic microwave background (CMB) and standard candles suggest it has the tiny value $\Lambda = (4.24 \pm 0.11) \times 10^{-66}\,\mathrm{eV}^2$ [24]: this presents an

[1] A 'minimal' smooth crossover [9] suggests $T_{\mathrm{c}} = (1.595 \pm 0.015) \times 10^{11}\,\mathrm{eV}$ [5], though many popular alternatives allow for a first-order phase transition [4, 6–8].

unresolved hierarchy within L_G, expressed as $\kappa\Lambda \sim 10^{-122}$. To avoid the hierarchy, it is tempting to reassign Λ to L_M. This however raises a severe fine-tuning problem, since the same value falls far short of conservative estimates[2] of the SM vacuum energy, dominated by bubbles of the Higgs, $W^{\pm}$ and Z bosons $\Lambda/\kappa\rho_{SM} \sim 10^{-53}$ [30, 31]. In practice this SM energy is simply neglected, while L_M is instead called upon to produce *cold dark matter* (CDM), a pressureless, phenomenological dust with no known couplings other than to gravity, and whose true origin within or beyond the SM (BSM) is obscure [32]. Dark matter also has broad phenomenological support on astrophysical scales, forming halos which support rapid galactic rotation rates [33, 34] and strong lensing [35] not otherwise explained by the luminous, baryonic matter of the SM. Finally, it is widely believed that LCDM should be supplemented by an inflationary model of the early Universe [36], which solves the horizon and flatness problems [37] while seeding cosmic perturbations with a nearly scale-invariant primordial power spectrum [38]. The mechanism of this inflation is not at all clear, but most models further augment L_M with *inflaton* degrees of freedom (D.o.F), occasionally with some SM provenance [39, 40].

Quite apart from its many theoretical loose-ends, LCDM is frequently suggested at the current time to now show signs of divergence from the findings of sophisticated cosmological survey experiments [41–47]. Foremost among these candidate discrepancies is the alleged *Hubble tension* [46, 47] between CMB-inferred 0.674 ± 0.005 [24] and locally-observed 0.735 ± 0.014 [48] determinations of the contemporary Hubble number $\mathsf{h} \equiv H_0/100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Cosmological tensions are not guaranteed to age well, but for the time being they fuel speculation over possible *new physics* and lend credence to long-standing programs which seek to replace the phenomenological patchwork with a *fundamental alternative to GR*.

Such alternatives are also motivated by theoretical loose-ends at the *shortest* length scales: as illustrated in Fig. 1, GR is perturbatively non-renormalisable [49]. Even at one loop, the inclusion of matter propagators spoils the renormalisability of pure GR by invoking quadratic curvature counter-terms which cannot be absorbed into the linear curvature invariant by rescaling [50–52]. A possible solution is to add such terms to the Einstein–Hilbert Lagrangian a priori. This approach culminates in the renormalisable theory of Stelle [53]. The addition of quadratic curvature invariants motivated in the ultraviolet (UV) should not interfere with the usual tests passed by GR in the infrared (IR). They do however necessarily result in higher-derivative theories whose unitarity may be questionable under standard quantisation schemes. For example, Stelle's theory contains a ghost in its tree-level graviton propagator, although recently this has been argued not to prevent unitary of the S-matrix [54]. Purely quadratic theories are not at all protected in the IR: the conformal gravity (CG) of Bach [55] also enjoys perturbative renormalisability and claims of unitary [56, 57] in the presence of a classical ghost [58], yet its phenomenological viability is subject to ongoing debate [59, 60].

One may also extend beyond quadratic invariants. The effective field theory (EFT) approach to gravity embraces (1) as the lowest order contribution to a power series in curvature, valid up to some context-dependent mass scale set by whichever part of L_T has been integrated out. This conservative approach facilitates a reliable and predictive theory of quantum gravity in a variety of scenarios, but forfeits the grail of UV completion [61, 62].

The problematic link between quadratic curvature additions and higher derivatives may be broken by reconsidering the fundamental dynamical variables of gravity. Such a route is suggested already within the tetrad formulation of GR, the minimal extension which facilitates gravitational coupling to the

[2] This particular figure is obtained by reasoning that Λ can be measured via the redshifting of supernovae light en route through an expanding Universe, i.e. a photon-graviton scattering process for which the relevant SM renormalisation scale is the geometric mean of the optical photon energy and the graviton energy set by the Hubble number.

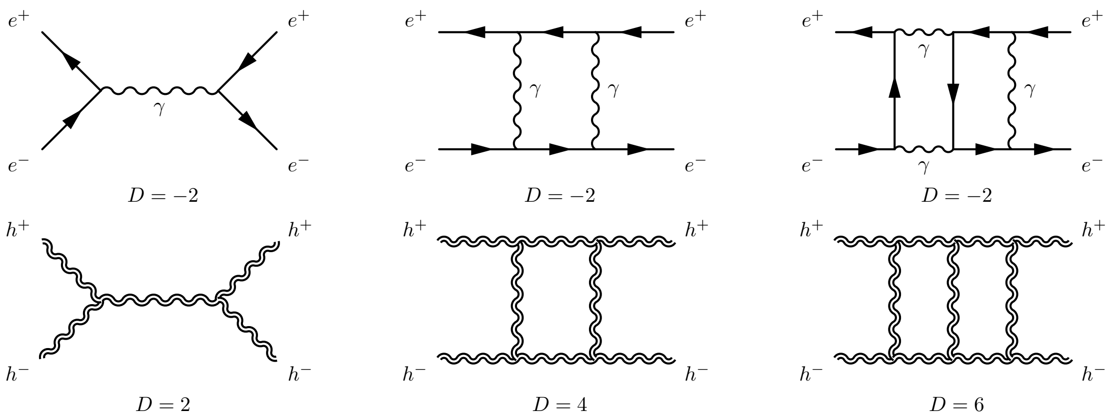

Fig. 1 Any amplitude in source-free GR diverges at sufficiently high perturbative order. For example, the superficial divergence D of diagrams contributing to the $hh\to hh$ scattering amplitude scales with loops and $(\partial h)^2h$ vertices. A perturbative renormalisation scheme for pure GR would therefore seem to require that infinitely many parameters be fixed by experiment. This is to be compared to renormalisable quantum electrodynamics (QED) in four dimensions, wherein D depends only on the external lines.

Dirac spinors of the SM (or indeed to *any* matter representation of $\mathrm{SL}(2,\mathbb{C})$ under the action of proper, orthochronous Lorentz rotations). By dint of their transformation under diffeomorphisms, the tetrads (vierbein) which split $g_{\mu\nu}$ can be interpreted as translational gauge fields $b^i{}_\mu$, where $g\sim b^2$. Higher derivatives are then eliminated if the Levi–Civita spin connection is promoted to a *wholly independent* gauge field $A^{ij}{}_\mu$ of the Lorentz group $\mathrm{SO}^+(1,3)$. The resulting *Poincaré gauge theory* (PGT) was pioneered by Kibble [63], Utiyama [64], Sciama [65] and others [66–69]. In terms of these new gauge potentials, the gravitational field strength tensors encode the torsion $\mathcal{T}\sim\partial b+bA$ and Riemann–Cartan curvature $\mathcal{R}\sim\partial A+A^2$. Being linear in the *first* derivatives of the gauge fields and the structure constants of the non-abelian (albeit non-compact) Poincaré group $\mathbb{R}^{1,3}\rtimes\mathrm{SO}^+(1,3)$, a theory quadratic in these tensors is close in spirit to the Yang–Mills $\mathrm{SU}(3)_\mathrm{c}$ and $\mathrm{SU}(2)_\mathrm{L}$ sectors of the SM. The Yang–Mills analogy could also be suggestive of the perturbative approach to renormalisation; popular alternatives exist, and some of these have Yang–Mills counterparts, too. For example, the asymptotic freedom of quantum chomodynamics (QCD) [70, 71] is sometimes mentioned with reference to hypothetical fixed points in the renormalisation group flow of GR [72, 73].

In this thesis we obtain by degrees a new alternative to (1) which is constructed entirely from quadratic invariants of the Riemann–Cartan curvature and the torsion[3]

$$\begin{aligned}L_\mathrm{G}=-\frac{4}{9\kappa}\mathcal{T}_i\mathcal{T}^i-\frac{\hat{\alpha}_6}{6}\Big[\Lambda\mathcal{T}_{ijk}\left(\mathcal{T}^{ijk}-2\mathcal{T}^{jik}\right)\\+\mathcal{R}_{ij}\big(\mathcal{R}^{[ij]}-12\mathcal{R}^{ij}\big)-2\mathcal{R}_{ijkl}\big(\mathcal{R}^{ijkl}-4\mathcal{R}^{ikjl}-5\mathcal{R}^{klij}\big)\Big]+2\hat{\alpha}_5\mathcal{R}_{[ij]}\mathcal{R}^{[ij]}.\end{aligned}\tag{2}$$

The Einstein–Hilbert term is thus replaced by the square of the torsion contraction $\mathcal{T}_i\equiv\mathcal{T}^j{}_{ij}$, whose dimensionful self-coupling strength is the Einstein constant κ. The cosmological constant Λ is revealed to be another torsion self-coupling: it is not introduced to parametrise a mysterious energy density. The constants $\hat{\alpha}_5$ and $\hat{\alpha}_6$ are dimensionless self-couplings of the Riemann–Cartan curvature, whose contractions are $\mathcal{R}\equiv\mathcal{R}^i{}_i\equiv\mathcal{R}^{ij}{}_{ij}$. Their values are not determined in the current work, but when the

[3] As a quick reference, our conventions for these quantities will be given in Eqs. (2.16) and (2.17) in Section 2.2.1.

conditions

$$\Lambda > 0, \quad \hat{\alpha}_6 < 0, \quad (\hat{\alpha}_5 + 2\hat{\alpha}_6)(\hat{\alpha}_5 - \hat{\alpha}_6) > 0, \tag{3}$$

are met, the linearisation of (2) on the matter-free, flat and torsionless background is both power-counting renormalisable (PCR) and unitary. In this regime two massless D.o.F and a massive pseudoscalar are propagated. If one insists on setting $\Lambda = 0$ in (2) only the last inequality of (3) is sufficient to guarantee these same properties, whereupon the pseudoscalar mass diverges and it becomes strongly coupled.

We will show that the theories (1) and (2) produce the same cosmology, but that in (2) this occurs because the axial vector torsion experiences a non-canonical kinetic stall at a constant vacuum expectation value (VEV). Initial (and optional) deviation from this VEV in the early Universe masquerades as *dark radiation* – though it introduces no new radiative D.o.F – and is advanced as a possible, partial solution to the Hubble tension, should such a tension persist.

We also obtain a variation of (2) in which the irreducible *tensor* part of the torsion is eliminated via multiplier fields ${\lambda^i}_{jk} \equiv {\lambda^i}_{[jk]}$

$$\begin{aligned} L_{\mathrm{G}} = -\frac{4}{9\kappa}\Big[\mathcal{T}_i\,\mathcal{T}^i + \lambda_{(ij)k}\big(2\mathcal{T}^{(ij)k} + \eta^{ij}\mathcal{T}^k\big)\Big] + \frac{\alpha_{\mathrm{CG}}}{18}\Big[\Lambda\mathcal{T}_{ijk}\,\big(\mathcal{T}^{ijk} - 2\mathcal{T}^{jik}\big) \\ + 3\mathcal{R}_{ij}\big(3\mathcal{R}^{[ij]} - 4\mathcal{R}^{ij}\big) - 2\mathcal{R}_{ijkl}\big(\mathcal{R}^{ijkl} - 4\mathcal{R}^{ikjl} - 5\mathcal{R}^{klij}\big)\Big]. \end{aligned} \tag{4}$$

This surgery, which we refer to as the 'tensor bypass', yields a theory whose dynamical structure is much simpler than that of (2): it *inherits* all the good phenomenology above, while determining $\hat{\alpha}_5$ to give a viable Newtonian limit around the torsion VEV. The VEV is acquired when the torsion emulates a *Cuscuton* field: unlike the Higgs field it lacks Goldstone's 'sombrero' potential, but it does break a conformal symmetry of the linear theory. Indeed the linear bypass theory happens to be identical to CG, complete with ghosts, allowing the remaining parameter $\hat{\alpha}_6$ to be interpreted as the conformal gravity coupling α_{CG}. We do not yet attempt an extension of this Newtonian limit to the full theory (2), or plumb the post-Newtonian regime in search of dark matter phenomenology. The general inhomogeneous environment of the torsionful vacuum in the full theory, its local particle content, fluctuations, stability and cosmological perturbation theory, are the natural focus of future study.

Our central result (2) is developed over Chapters 2 and 3. In Chapter 2 we consider the cosmology of unitary, PCR PGT, and of its scale-invariant extensions which gauge the Weyl group $\mathrm{W}(1,3)$. In Chapter 3 we construct a torsionless bi-scalar-tensor theory which represents the general cosmology, revealing a generally non-canonical kinetic structure buried in the gauge theory. Extensions culminating in (4) are developed over Chapters 4 and 5. In Chapter 4 we explore the Hamiltonian structure of unitary, PCR PGT, and identify concerning dynamical features. In Chapter 5 we attempt to address these issues by formulating a general multiplier-constrained version of PGT. Chapter 1 acts as an independent prelude to the main body of the thesis, and concerns only GR and its minimal gauge theory extensions. In particular we focus on a very peculiar aspect of GR; that although there is no doubt that gravitational fields store and convey energy, it is impossible to say *where* in spacetime that energy is located.

Chapter 1

Localising the energy of static Einstein–Hilbert theory

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1.1 Introduction

It is widely accepted that the energy contained in a gravitational field cannot in general be localised. This paradigm, which developed over the century following the advent of GR[1], is often regarded as a consequence of the equivalence principle. It is equally well accepted that gravitational fields carry such an energy in the first place. The ebb and flow of energy-momentum between matter and gravity explains the emission and recent detection of gravitational waves, with both processes mediated by the covariant conservation of the stress-energy tensor (SET) of matter, or from (1) in a tensor slot notation

$$\nabla_{\mathfrak{a}} T^{\mathfrak{a}}_{\mathfrak{b}} = 0, \quad T_{\mathfrak{ab}} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{M}}}{\delta g^{\mathfrak{ab}}}, \quad S_{\mathrm{M}} \equiv \int \epsilon_{\mathfrak{abcd}} L_{\mathrm{M}}, \quad g \equiv \det g_{\mathfrak{ab}}. \tag{1.1}$$

Counter intuitively, this law states that the energy and momentum of matter, whose densities are the contractions of $T^{\mathfrak{a}}_{\mathfrak{b}}$ with an observer's four-velocity, are *not* generally conserved in the presence of a gravitational field. Many introductory texts (e.g. [15, 75]) are quick to observe such energy-momentum exchange is not inevitable through the generic counterexample: a stationary spacetime is furnished with a global timelike Killing vector $K^{\mathfrak{a}}$, so the quantity

$$Q_{\mathrm{T}} \equiv \int_{\Sigma_t} \epsilon_{\mathfrak{abcd}} T^{\mathfrak{ae}} K_{\mathfrak{e}}, \tag{1.2}$$

is independent of the Cauchy hypersurface Σ_t used to define it, and hence constitutes a conserved charge.

[1] For a comprehensive discussion of the 'energy problem' in GR, see for example the review by J. Goldberg [74].

In contrast to energy localisation, the global picture of gravitational energetics is often less ambiguous. This is the case for many spacetimes of astrophysical interest, which can be 'patched on' to the Universe at large because they are asymptotically flat. In such cases the Newtonian regime at spatial infinity provides an observer with a clear account of the total gravitational mass M_{T} of the system. Komar [76] proposed a derived quantity for stationary, asymptotically flat systems, designed to agree with precisely this Newtonian value. The integral theorem can be used to obtain a Komar mass 'density' which is proportional to the Ricci tensor. Such a picture, in which M_{T} is exclusively distributed wherever $T^{\mathfrak{ab}}$ is nonvanishing, is dissatisfying because it does not reflect the common assumption that part of the gravitational mass is locked away in the gravitational field, a view explored more recently by Katz, Lynden-Bell and Bičák [77, 78]. The notion of asymptotic flatness was developed during the 'golden age' of GR to imply conformal isometry to a bounded region of some curved, nonphysical spacetime (in the case of Minkowski spacetime, which has asymptotic flatness, the nonphysical spacetime is Einstein's static Universe). This enabled the mass of Komar to be generalised to that of Bondi [79] which is evaluated at various sections of null infinity[2]. At spatial infinity, the Hamiltonian formulation of GR attributed to Arnowitt, Deser and Misner (ADM) [80] provides a further definition of M_{T} reliant on asymptotically Cartesian coordinates.

For those dissatisfied with the global picture, attempts to localise gravitational energetics usually take the form of energy-momentum *complexes* – objects of questionable gauge invariance which emulate a combined SET for matter and gravity. A basic requirement of a complex following from this discussion is that it integrates in some sense to give M_{T}. A further requirement is that it be identically conserved, as with the matter SET in special relativity. Conservation is built in by defining the complex to be the gradient of a superpotential constructed from the dynamical variables of gravity. The first complex ${}_{\mathrm{E}}\theta_{\mu}{}^{\nu}$ was proposed by Einstein in 1916 [81], though the corresponding superpotential is attributed to Freud [82]

$$ {}_{\mathrm{E}}\theta_{\mu}{}^{\nu} \equiv \partial_{\lambda}\, {}_{\mathrm{F}}\Psi_{\mu}{}^{\nu\lambda}, \tag{1.3} $$

where Freud's superpotential ${}_{\mathrm{F}}\Psi_{\mu}{}^{\nu\lambda}$ is a function of the metric and its first derivatives and is skew-symmetric in its final pair of indices. The utility of the 'special' conservation law

$$ \partial_{\nu}\, {}_{\mathrm{E}}\theta_{\mu}{}^{\nu} \equiv 0, \tag{1.4} $$

is evident when the field equations are used to collect the second derivatives of the metric appearing in ${}_{\mathrm{E}}\theta_{\mu}{}^{\nu}$ into T^{ν}_{μ}, thereby partitioning the energetics of matter and gravity

$$ {}_{\mathrm{E}}\theta_{\mu}{}^{\nu} = \sqrt{-g}(T^{\nu}_{\mu} + {}_{\mathrm{E}}t_{\mu}{}^{\nu}). \tag{1.5} $$

The remaining quantity ${}_{\mathrm{E}}t_{\mu}{}^{\nu}$ is known as *Einstein's pseudotensor*. Over the four decades following the introduction of (1.5), the evident freedom in the choice of superpotential led many authors to develop their own complexes, including Landau and Lifshitz [75], Komar [76, 83] and Møller [84, 85]. During this time, the scope of the geometric theory of gravitation was expanded from the GR of Riemann spacetime V_4, to the Einstein–Cartan theory (ECT) of Riemann–Cartan space U_4. This resulted in the title of 'dynamical variable of gravity' passing from the metric to its 'square root', which is the tetrad or vierbein.

[2] Bondi's mass will not be of use to us because we do not consider radiating systems.

Having encountered difficulties with metrical attempts, Møller [84] constructed a new superpotential ${}_{\mathrm{M}}\Psi_{\mu}{}^{\nu\lambda}$ from the tetrad and its first derivatives. Unlike its predecessors, Møller's superpotential is a tensor under the usual diffeomorphisms of the spacetime, but not under Lorentz rotations of the tetrads. The corresponding energy-momentum complex

$$ {}_{\mathrm{M}}\theta_{\mu}{}^{\nu} = \sqrt{-g}(T^{\nu}_{\mu} + {}_{\mathrm{M}}t_{\mu}{}^{\nu}), \tag{1.6} $$

and pseudotensor are otherwise fairly analogous to those of Einstein.

In the context of these opening remarks, it is not surprising that energy-momentum complexes suffer greatly under the principle of equivalence. Many authors have objected that in a falling frame, the associated pseudotensors promptly vanish along with their local account of gravitational energy. Consequently their deployment is usually confined to privileged quasi-Cartesian coordinate systems, though this is at least compatible with the techniques of the Hamiltonian formulation at spatial infinity.

Just as the Newtonian regime provides a valuable global concept of gravitational energy, so it has proven useful in energy localisation: from the perspective of linear gravity on a flat background, pseudotensors and tensors are indistinguishable. Bičák and Schmidt [86] have charted the freedom and ambiguity that is to be found at lowest perturbative order in the construction of gravitational SETs. Their analysis includes a symmetric tensor ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$ developed in a recent paper by Butcher, Hobson and Lasenby [87]. If the physical spacetime $\mathcal{M}$ is an example of V_4, this tensor is constructed in the background spacetime $\check{\mathcal{M}}$ (which is Minkowski space M_4), to account for the local non-conservation of matter energy-momentum implied by (1.1). In the harmonic gauge, it is the unique symmetric tensor to do so. A curious observation made in [87] is that ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$ treats the Newtonian gravitational potential as if it were a matter-generated Klein–Gordon field. In [88] a similar procedure led to a tensor for gravitational spin. The same authors demonstrated in [89] that these tensors are the canonical Noether currents in ECT under a perturbative expansion of the Einstein–Cartan Lagrangian which they developed in [90].

The succession of ECT was (and largely remains) formal, with the vast majority of the literature addressing GR. Nevertheless, the advent of the tetrad and spin connection eventually gave rise to a rich new class of *gauge theories* of gravity. The Poincaré group was fully gauged by Kibble[3] who considered an action analogous to that of Einstein and Hilbert in which the gravitational gauge fields are minimally coupled to matter. The unifying mathematical language of geometric algebra has also been used to gauge the geometric algebra of Minkowski spacetime, or *spacetime algebra* (STA). During the procedure, there arise natural ways to implement minimal coupling and an Einstein–Hilbert action. The result [69, 93], known as *gauge theory gravity* (GTG), may be re-interpreted geometrically[4] as ECT in which torsion is sourced by material spin.

There are two common alternative approaches to constructing energy-momentum complexes. Rather than composing the relevant superpotential from the beginning, it may prove efficient to isolate it by 'splitting' the Einstein equations. Hestenes [94] has demonstrated that this method lends itself to GTG in the STA, where the Einstein tensor can be written in his *unitary form*. Accordingly he obtains the complexes (or splits) of Einstein, Landau–Lifshitz and Møller, along with one which is original. The other method is that referred to by Møller as *variational*: it may be possible to construct an alternative Lagrangian to that of Einstein and Hilbert, from which the required energy-momentum

[3] Kibble's work [91] follows early efforts by Utiyama and concurrently with Sciama – see Section I of [92] and the references therein.

[4] Modulo topological effects.

complex follows as an (affine) canonical SET. Møller employed both variational and superpotential methods when proposing (1.6), whilst the variational approach was suggested by Einstein for (1.5). Unlike the geometric theories of Møller's day, GTG was developed a priori as a Lagrangian field theory, so one would expect it to be well suited to the variational method.

An important feature of gravitational gauge theories is that gravitational energy can be defined via Noether theorems, in a way which is known to map on to the Hamiltonian formalism [95]. This has been applied in particular to the PGT in [96], where the various pseudotensor statements of energy are interpreted as values of the Hamiltonian with various boundary conditions. We will not consider these methods further here: the PGT is properly introduced in Chapter 2, and its Hamiltonian formulation[5] in Chapter 4.

In this chapter we use the methods of GTG and the STA to provide a fresh perspective on the localisation of M_{T} and the rôle of the conserved charge Q_{T}. We confine our discussion to static spacetimes containing perfect fluids without spin. In the ensuing absence of torsion, GTG may be geometrically interpreted as GR. In our treatment, we will relate the formalisms of Butcher, Einstein and Møller.

The remainder of this chapter is set out as follows. Section 1.2 pertains to GR. In Section 1.2.1 through to Section 1.2.4 we review the approach of Butcher and see how it might be extended to nonlinear gravity. In Section 1.2.5 we make some observations on the relativistic mass of static, spherically symmetric perfect fluids.

Section 1.3 addresses some of the issues raised in Section 1.2 using the gauge theory approach, beginning with a brief introduction to GTG. In Sections 1.3.1 and 1.3.2 we discuss energy localisation formalisms in GTG as obtained through the variational approach, in particular the pseudotensor of Møller. This enables us to generalise the Klein–Gordon correspondence of ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$ in Section 1.3.3, and gives us a new perspective on the mass of Komar in Section 1.3.4. Conclusions follow in Section 1.4. Fraktur Roman letters label Penrose's abstract indices [15] which are necessary to connect with Butcher's formalism within this chapter, after which we transition fully to Greek coordinate indices. Using the latter, our conventions for the Riemann tensor and the Christoffel symbols will be

$$R_{\rho\sigma\mu}{}^{\nu} \equiv 2\big(\partial_{[\sigma}\Gamma^{\nu}_{\rho]\mu} + \Gamma^{\lambda}_{[\rho|\mu}\Gamma^{\nu}_{|\sigma]\lambda}\big), \quad \Gamma^{\mu}_{\nu\sigma} \equiv \frac{1}{2}g^{\mu\lambda}\big(\partial_{\nu}g_{\sigma\lambda} + \partial_{\sigma}g_{\nu\lambda} - \partial_{\lambda}g_{\sigma\nu}\big), \tag{1.7}$$

with the Ricci tensor $R^{\mu}_{\nu} \equiv R^{\mu\sigma}{}_{\nu\sigma}$ and scalar $R \equiv R^{\mu}_{\mu}$[6].

1.2 Energy-momentum and mass in GR

1.2.1 Previous work on the flat background

We begin by providing an introduction to the formalism of [87]. This work is grounded in the mapping $\phi : \mathcal{M} \to \check{\mathcal{M}}$ from the physical spacetime $\mathcal{M}$, a Riemann space V_4 with metric $g_{\mathfrak{ab}}$, to a flat background $\check{\mathcal{M}}$, a Minkowski space M_4 with metric $\check{g}_{\mathfrak{ab}}$. The background is furnished with four Lorentzian coordinate functions $\{x^{\mu}\}$ labelled by Greek indices, so that in $\check{\mathcal{M}}$ the vector $\partial/\partial x^{\mu}$ has components $(\check{\boldsymbol{e}}_{\mu})^{\mathfrak{a}}$ which

[5] Note that a new variational version of the Hamiltonian formulation of PGT was recently proposed in [97].

[6] In Chapter 2 we will promote this quantity to the Riemann–*Cartan* tensor $\mathcal{R}_{ijk}{}^{l}$, whose Ricci contraction $\mathcal{R}_{i}{}^{l}$ is written with staggered indices to reflect the loss of symmetry induced by torsion.

obey $(\check{\boldsymbol{e}}_\mu)^{\mathfrak{a}}(\check{\boldsymbol{e}}_\nu)^{\mathfrak{b}}\check{g}_{\mathfrak{ab}} \equiv \eta_{\mu\nu}$ and $\check{\nabla}_{\mathfrak{a}}(\check{\boldsymbol{e}}_\mu)^{\mathfrak{b}} = 0$. In $\mathcal{M}$ the image coordinate functions $\{y^\mu\}$ are formed by the pullback of the x^μ at any $p \in \mathcal{M}$: $y^\mu(p) = \phi_*(x^\mu)(p) = x^\mu \circ \phi(p)$. From these image coordinates a new basis $\partial/\partial y^\mu$ has components $(\phi^*\boldsymbol{e}_\mu)^{\mathfrak{a}} = (\check{\boldsymbol{e}}_\mu)^{\mathfrak{a}}$. Crucially, whilst the $(\check{\boldsymbol{e}}_\mu)^{\mathfrak{a}}$ are trivially components of Killing vectors in $\check{\mathcal{M}}$, the same is not generally true of the $(\boldsymbol{e}_\mu)^{\mathfrak{a}}$ in $\mathcal{M}$ because of the presence of the gravitational field. In the physical spacetime this leads to the non-conservation of the four local four-current densities of matter energy-momentum $J_\mu{}^{\mathfrak{a}}$ which are formed by contracting $T_{\mathfrak{ab}}$ with the new basis vectors

$$\nabla_{\mathfrak{a}} J_\mu{}^{\mathfrak{a}} = T^{\mathfrak{a}}_{\mathfrak{b}} \nabla_{\mathfrak{a}}(\boldsymbol{e}_\mu)^{\mathfrak{b}} \neq 0. \tag{1.8}$$

According to the conventional perturbation scheme

$$\phi^* g_{\mathfrak{ab}} = \check{g}_{\mathfrak{ab}} + h_{\mathfrak{ab}}, \quad \phi^* g^{\mathfrak{ab}} = \check{g}^{\mathfrak{ab}} - h^{\mathfrak{ab}} + \mathcal{O}(h^2), \tag{1.9}$$

the method of [87] is to cancel this matter stress-energy 'leak' as it is manifest in a flat background with the equal and opposite covariant divergence of some tensor ${}_{\text{B}}\tau \sim \check{\nabla}h\check{\nabla}h$. This tensor is thus determined by the background relation

$$\check{\nabla}^{\mathfrak{a}}{}_{\text{B}}\tau_{\mathfrak{ad}} = -\left[\phi^*\left(T^{\mathfrak{a}}_{\mathfrak{b}}\nabla_{\mathfrak{a}}(\boldsymbol{e}_\mu)^{\mathfrak{b}}\right)(\check{\boldsymbol{e}}^\mu)_{\mathfrak{d}}\right]^{(2)}, \tag{1.10}$$

where objects to nth order in h will be identified with a parenthesised superscript. Remarkably, a symmetric superpotential-free ansatz for ${}_{\text{B}}\tau_{\mathfrak{ab}}$ combined with the harmonic gauge constraint

$$\check{\nabla}^{\mathfrak{a}}\bar{h}_{\mathfrak{ab}} = 0, \quad \bar{h}_{\mathfrak{ab}} \equiv h_{\mathfrak{ab}} - \frac{1}{2}\eta_{\mathfrak{ab}}h, \quad h \equiv h^{\mathfrak{a}}_{\mathfrak{a}} \tag{1.11}$$

was found to yield a unique solution to (1.10)

$$\kappa_{\text{B}}\bar{\tau}_{\mathfrak{de}} \equiv \frac{1}{4}\check{\nabla}_{\mathfrak{d}}h_{\mathfrak{ab}}\check{\nabla}_{\mathfrak{e}}\bar{h}^{\mathfrak{ab}}. \tag{1.12}$$

Note that the overbar notation when dealing with tensors signifies the trace-reverse, and has a different meaning in the geometric algebra used in Section 1.3.

1.2.2 A non-linear generalisation

As we established above, the tensor ${}_{\text{B}}\tau_{\mathfrak{ab}}$ is defined in $\check{\mathcal{M}}$, and we would like to augment it with higher order corrections in h. We can invent a new tensor for the full series

$$\check{\mathcal{T}}_{\mathfrak{ab}} \equiv \sum_{n=2}^{\infty} \check{\mathcal{T}}^{(n)}{}_{\mathfrak{ab}} \stackrel{(?)}{=} {}_{\text{B}}\tau_{\mathfrak{ab}} + \sum_{n=3}^{\infty} \check{\mathcal{T}}^{(n)}{}_{\mathfrak{ab}}, \tag{1.13}$$

anticipating $\check{\mathcal{T}}_{\mathfrak{ab}}$ to be the pushforwards of some $\mathcal{T}_{\mathfrak{ab}}$ in $\mathcal{M}$. A natural extension of the theory to third perturbative order is to introduce the ansatz $\check{\mathcal{T}}^{(3)} \sim h\check{\nabla}h\check{\nabla}h$ to the equation

$$\check{\nabla}^{\mathfrak{a}}\check{\mathcal{T}}^{(3)}{}_{\mathfrak{ad}} \stackrel{(?)}{=} -\left[\phi^*\left(T^{\mathfrak{a}}_{\mathfrak{b}}\nabla_{\mathfrak{a}}(\boldsymbol{e}_\mu)^{\mathfrak{b}}\right)(\check{\boldsymbol{e}}^\mu)_{\mathfrak{d}}\right]^{(3)}, \tag{1.14}$$

however after a long calculation we find that this *has no solution* under (1.9) with or without the harmonic gauge condition[7]. One way to proceed is to generalize the form of the background covariant derivative on the LHS of (1.14) by introducing some 'friction connection' of the form $F^{(1)} \sim \check{\nabla} h$ which couples to ${}_{\mathrm{B}}\tau_{\mathfrak{a}\mathfrak{b}}$. This connection will be constructed so as to account for the non-conservation as it appears even in $\check{\mathcal{M}}$. Accordingly, the third-order correction must instead obey

$$\check{\nabla}^{\mathfrak{a}}\check{\mathcal{T}}^{(3)}{}_{\mathfrak{a}\mathfrak{d}} - \check{g}^{\mathfrak{a}\mathfrak{f}}\left(F^{(1)\mathfrak{c}}{}_{\mathfrak{f}\mathfrak{a}}\,{}_{\mathrm{B}}\tau_{\mathfrak{c}\mathfrak{d}} + F^{(1)\mathfrak{c}}{}_{\mathfrak{d}\mathfrak{a}}\,{}_{\mathrm{B}}\tau_{\mathfrak{f}\mathfrak{c}}\right) = -\left[\phi^{*}\left(T^{\mathfrak{a}}_{\mathfrak{b}}\nabla_{\mathfrak{a}}(\boldsymbol{e}_{\mu})^{\mathfrak{b}}\right)(\check{\boldsymbol{e}}^{\mu})_{\mathfrak{d}}\right]^{(3)}. \tag{1.15}$$

This turns out to be very fruitful. The ansatz for $\check{\mathcal{T}}^{(3)}{}_{\mathfrak{a}\mathfrak{b}}$ cannot be solved uniquely, but it gives a space of *asymmetric* third-order corrections to ${}_{\mathrm{B}}\tau_{\mathfrak{a}\mathfrak{b}}$. More importantly, by repeating the procedure at higher orders it quickly becomes apparent that the $F^{(n)\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}$ are in fact terms from the perturbative expansion in h of the Levi–Civita connection, familiar in torsion-free GR as the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$. Things now become clearer: the apparent 'friction' in the background is nothing more than curvature creeping into the theory at higher orders. Because the Levi–Civita connection is a function of the gradient of the metric, it makes its first appearance in the third-order equation, (1.15), spoiling the flat-space picture as it does so: the quadratic ${}_{\mathrm{B}}\tau_{\mathfrak{a}\mathfrak{b}}$ is a special case that makes [87] possible.

It is now easy to extend the theory to all orders. We aim to soak up the matter stress-energy leak directly in the physical spacetime with the equal and opposite covariant divergence of some *gravitational* stress-energy four-currents $\mathcal{J}_{\mu}{}^{\mathfrak{a}}$. We will suppose these currents to be formed from some tensor $\mathcal{J}_{\mu}{}^{\mathfrak{a}} \equiv \mathcal{T}^{\mathfrak{a}}_{\mathfrak{b}}\,(\boldsymbol{e}_{\mu})^{\mathfrak{b}}$, to be identified as a gravitational SET. Note in particular that so long as $\mathcal{T}_{\mathfrak{a}\mathfrak{b}}$ has Penrose indices we do mean it to be a tensor-valued object, which may be covariantly differentiated to give

$$\nabla_{\mathfrak{a}}\mathcal{J}_{\mu}{}^{\mathfrak{a}} = \nabla_{\mathfrak{a}}\left(\mathcal{T}^{\mathfrak{a}}_{\mathfrak{b}}\,(\boldsymbol{e}_{\mu})^{\mathfrak{b}}\right) = -\nabla_{\mathfrak{a}}J_{\mu}{}^{\mathfrak{a}} = -\frac{1}{\kappa}\nabla_{\mathfrak{a}}\left(G^{\mathfrak{a}}_{\mathfrak{b}}(\boldsymbol{e}_{\mu})^{\mathfrak{b}}\right), \quad G_{\mathfrak{a}\mathfrak{b}} \equiv R_{\mathfrak{a}\mathfrak{b}} - \frac{1}{2}g_{\mathfrak{a}\mathfrak{b}}R. \tag{1.16}$$

In the last equality, the matter SET is translated into curved spacetime using the Einstein equations. In general, the only such tensor that satisfies (1.16) is proportional to the Einstein tensor itself[8]. To distance ourselves from this fact, notice that (1.16) takes on a tidier form in the $\{y^{\mu}\}$ coordinate system. In this case, the components of the basis are given by the Kronecker delta, so (1.16) reduces to

$$\kappa\left(\partial_{\mu}\mathcal{T}^{\mu}{}_{\lambda} + \Gamma^{\mu}_{\nu\mu}\mathcal{T}^{\nu}{}_{\lambda}\right) = -G^{\mu}_{\nu}\Gamma^{\nu}_{\mu\lambda}. \tag{1.17}$$

We want $\mathcal{T}_{\mu\nu}$ to be second-order in the first derivatives of the metric. Using therefore the ansatz $\mathcal{T} \sim \partial g \partial g$ it can be shown that (1.17) has a *unique*[9] solution without the need for further gauge constraints (such as the harmonic coordinate condition). It may be written compactly in trace-reversed form as the following function of spacetime:

$$\kappa\bar{\mathcal{T}}_{\sigma\lambda} \equiv \frac{1}{4}g^{\nu\kappa}g^{\rho\mu}\left(\partial_{\sigma}g_{\rho\nu}\partial_{\lambda}g_{\mu\kappa} - \partial_{\sigma}g_{\rho\mu}\partial_{\lambda}g_{\nu\kappa} + \partial_{\mu}g_{\sigma\rho}\partial_{\lambda}g_{\nu\kappa} + \partial_{\mu}g_{\nu\kappa}\partial_{\lambda}g_{\sigma\rho} - 2\partial_{\nu}g_{\sigma\rho}\partial_{\lambda}g_{\mu\kappa}\right). \tag{1.18}$$

1.2.3 Einstein's pseudotensor

Two sinister features of the function (1.18) are immediately obvious: firstly it is asymmetric in its indices and secondly it emphatically does *not* constitute a tensor definition for some $\mathcal{T}_{\mathfrak{a}\mathfrak{b}}$. These features are

[7] Neither this nor the asymmetry of the solution to (1.15) will be alleviated by the 'central expansion' mentioned in [89].
[8] In Section 1.3.1 we will explore the gauge theory version of this 'tautology'.
[9] The calculation is longer than that required to obtain ${}_{\mathrm{B}}\tau_{\mathfrak{a}\mathfrak{b}}$ in [87], but takes a similar form.

explained by a third observation, that (1.18) are identically the components of the transposed[10] Einstein pseudotensor ${}_{\mathrm{E}}t_{\mu}{}^{\nu}$, in the $\{y^{\mu}\}$ coordinate system $\mathcal{T}_{\sigma\lambda} = {}_{\mathrm{E}}t_{\lambda\sigma}$.

In hindsight it is easy to see why we have arrived at the oldest description of gravitational energetics in GR. We saw in Section 1.1 that Einstein's energy-momentum complex admits a special conservation law. Given the partitioning in (1.5) this law becomes $\partial_{\nu}\left(\sqrt{-g}{}_{\mathrm{E}}t_{\mu}{}^{\nu}\right) = -\partial_{\nu}\left(\sqrt{-g}T^{\nu}_{\mu}\right)$. If we cast it in the $\{y^{\mu}\}$ coordinates and differentiate the metric determinant according to $\partial_{\lambda}\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_{\lambda}g_{\mu\nu} = \sqrt{-g}\Gamma^{\mu}_{\lambda\mu}$, we are left (once the indices on $\mathcal{T}_{\mu\nu}$ are swapped) with precisely the motivating equation (1.17), which is the generalisation of the local conservation law of [87]. Whilst ${}_{\mathrm{E}}t_{\mu\nu}$ is the same quadratic function of the metric derivatives in all coordinate systems (i.e. is a pseudotensor), $\mathcal{T}_{\mathfrak{ab}}$ was set up as a tensor that coincides with ${}_{\mathrm{E}}t_{\mathfrak{ab}}$ in the $\{y^{\mu}\}$ coordinate system. As with the pseudotensor of Landau and Lifshitz, it is possible to write ${}_{\mathrm{E}}t_{\mu\nu}$ as a quadratic function of the $\Gamma^{\lambda}_{\mu\nu}$, so $\mathcal{T}_{\mathfrak{ab}}$ can be constructed as a quadratic function of the $\check{\nabla}_{\mathfrak{b}}(\boldsymbol{e}_{\mu})^{\mathfrak{a}}$ and $\check{\nabla}_{\mathfrak{b}}(\boldsymbol{e}^{\mu})_{\mathfrak{a}}$, using the $(\boldsymbol{e}_{\mu})^{\mathfrak{a}}$ and $(\boldsymbol{e}^{\mu})_{\mathfrak{a}}$ to contract away all Lorentz indices. This does not help however, because we have only recast a pseudotensor as a tensor-valued function of a privileged coordinate system.

In the harmonic gauge and to lowest order[11], $\check{\mathcal{T}}_{\mathfrak{ab}}$ and ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$ do not agree, since

$$\kappa\bar{\check{\mathcal{T}}}^{(2)}{}_{\mathfrak{fd}} = \frac{1}{4}\check{\nabla}_{\mathfrak{f}}h_{\mathfrak{ab}}\check{\nabla}_{\mathfrak{d}}\bar{h}^{\mathfrak{ab}} + \frac{1}{4}\check{\nabla}_{\mathfrak{a}}h\check{\nabla}_{\mathfrak{d}}h^{\mathfrak{a}}{}_{\mathfrak{f}} - \frac{1}{2}\check{\nabla}_{\mathfrak{a}}h_{\mathfrak{fb}}\check{\nabla}_{\mathfrak{d}}h^{\mathfrak{ab}}, \tag{1.19}$$

which differs from $\kappa_{\mathrm{B}}\bar{\tau}_{\mathfrak{ab}}$ by the last two terms[12]. There is no contradiction here because so long as the harmonic condition holds it can be shown that $\check{\nabla}^{\mathfrak{a}}\left(\check{\mathcal{T}}^{(2)}{}_{\mathfrak{af}} - {}_{\mathrm{B}}\tau_{\mathfrak{af}}\right) = 0$. Therefore, the tensor ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$ is formed by trimming an identically conserved quantity (a 'gauge current') from the linearised Einstein pseudotensor in the harmonic gauge.

1.2.4 The linearised Klein–Gordon correspondence

The tensor ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$ lends itself well to the Newtonian limit of gravitostatics. An inertial observer with velocity u^{μ} near a perfect fluid in hydrostatic equilibrium finds the matter SET to be $\check{T}_{\mu\nu} = \rho u_{\mu}u_{\nu}$ to lowest order in h (neglecting pressure). The linearised field equations in the harmonic gauge $\check{\Box}\bar{h}_{\mathfrak{ab}} = -2\kappa\check{T}_{\mathfrak{ab}}$ yield the familiar Newtonian potential $h_{\mu\nu} = 2\varphi(2u_{\mu}u_{\nu} - \eta_{\nu\mu})$, which obeys

$$\check{\Box}^{2}\varphi = -\kappa\rho/2. \tag{1.20}$$

To give a minimal example, a compact spherically symmetric distribution of total mass M_{T} gives rise to the external potential $\varphi = -\kappa M_{\mathrm{T}}/8\pi r$ and metric perturbation

$$\mathrm{d}s^{2} = (1 - \kappa M_{\mathrm{T}}/4\pi r)\,\mathrm{d}t^{2} - (1 + \kappa M_{\mathrm{T}}/4\pi r)\,\mathrm{d}\mathbf{x}^{2}, \tag{1.21}$$

where $\mathrm{d}\mathbf{x}^{2} \equiv \mathrm{d}x^{\alpha}\mathrm{d}x^{\alpha}$ for $\alpha \in \{1, 2, 3\}$. This is the Newtonian limit of the *rectangular isotropic* line element for Schwarzschild spacetime: we therefore see that isotropic coordinates arise naturally in linear

[10] Because of the natural definition of gravitational energy-momentum currents in [87], the extension $\mathcal{T}_{\mu\nu}$ is defined with a transpose relative to the pseudotensor of Einstein.

[11] Note if a pseudotensor becomes an affine tensor at lowest order, Penrose indices may be temporarily used.

[12] So, the series (1.13) was only nearly correct.

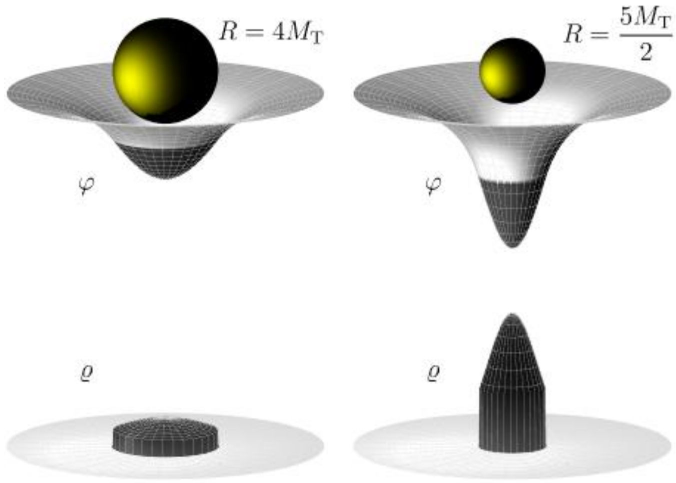


Fig. 1.1 The picture of energetics to be developed in Section 1.3.3 using isotropic coordinates in the gauge theory approach. The pseudotensors of Møller and Einstein both describe gravitational stress-energy as if the gravitational potential φ were a real Klein–Gordon field, generated by a source density ϱ, which in turn integrates to give the gravitational mass of the system M_T. Here, for a pair of highly relativistic Schwarzschild stars approaching collapse to a black hole, the Newtonian form of the potential $\varphi = -\kappa M_T/8\pi r$ is preserved right down to the stellar surface. We have already seen in Section 1.2.4 how this 'Klein–Gordon correspondence' is reflected by the tensor of Butcher in the Newtonian limit.

gravitostatics. Another way to arrive at the field equation (1.20) is through the Klein–Gordon theory[13]

$$L_{\text{KG}} \equiv \frac{1}{\kappa} X^{\varphi\varphi} - \varphi\rho, \quad X^{\varphi\varphi} \equiv \frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi, \tag{1.22}$$

which highlights a curious feature of ${}_{\text{B}}\tau_{\mathfrak{ab}}$. Since $\kappa_{\text{B}}\bar{\tau}_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi$, we see that ${}_{\text{B}}\tau_{\mu\nu}$ is describing the stress and energy bound up in the Newtonian potential as if it were a scalar field generated by matter. At linear order there is room for this 'Klein–Gordon correspondence' to appear coincidental, but equipped with the generalisation to Einstein's pseudotensor, we will show in Section 1.3.3 that the principle does in fact apply at all orders – this is illustrated in Fig. 1.1 below for a pair of Schwarzschild stars with the same gravitational mass but different densities. To make this generalisation, we will need not only isotropic coordinates, but some sense of the flat background $\check{\mathcal{M}}$ in nonlinear gravity. This construct is provided naturally by the gauge theory approach.

1.2.5 Mass in GR

Before addressing energy localisation in the gauge theory approach, we will make some observations regarding relativistic mass. The spacetimes of particular interest to us will be not only stationary but static. Consequently there will be a global timelike Killing vector $K^{\mathfrak{a}}$, such that if a Cauchy surface Σ_t were defined to be a contour of the Killing parameter t, $K^{\mathfrak{a}}$ would be orthogonal to that

[13] The Klein–Gordon field here is real, but for convenience the kinetic term lacks a factor of 1/2, in common with the complex (charged) theory. More in keeping with the various conventions of this thesis, especially Chapter 3, we could equally consider the theory $L_{\text{KG}} \equiv X^{\phi\phi} - \phi\rho/\sqrt{2}m_{\text{p}}$ for $X^{\phi\phi} \equiv \frac{1}{2} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ and $\phi \equiv \sqrt{2}m_{\text{p}}\varphi$.

surface. Furthermore the spacetimes will be asymptotically flat in the sense discussed above, spherically symmetric and regular everywhere. This restricts us to precisely those spacetimes which were of earliest astrophysical interest, since they accommodate non-spinning relativistic stars. We will consider stars composed of a perfect fluid. It is convenient to study these systems using *Schwarzschild-like* coordinates, with general line element

$$\mathrm{d}s^2 = e^A \mathrm{d}t^2 - e^B \mathrm{d}\bar{r}^2 - \bar{r}^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2). \tag{1.23}$$

These coordinates have the advantage of preserving the ratio of 2π between the radial coordinate $\bar{r}$ and the proper distance about the equator. Less frequently used (we have seen these already in Section 1.2.4 and will use them extensively in Section 1.3.3) are the *isotropic* coordinates

$$\mathrm{d}s^2 = e^A \mathrm{d}t^2 - e^C[\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2)]. \tag{1.24}$$

The SET of a perfect fluid with proper density ρ, pressure P and bulk four-velocity $u^{\mathfrak{a}}$ is

$$T^{\mathfrak{a}\mathfrak{b}} = (\rho + P)\, u^{\mathfrak{a}} u^{\mathfrak{b}} - P g^{\mathfrak{a}\mathfrak{b}}. \tag{1.25}$$

From the Einstein equations, we will require

$$\kappa\rho = B' e^{-B}/\bar{r} + (1 - e^{-B})/\bar{r}^2, \quad \kappa P = A' e^{-B}/\bar{r} - (1 - e^{-B})/\bar{r}^2, \tag{1.26}$$

where throughout this section, prime denotes differentiation with respect to $\bar{r}$. Some very useful derived results are then

$$A' = \frac{\kappa M \left(1 + 4\pi\bar{r}^3 P/M\right)}{4\pi\bar{r}^2\left(1 - \kappa M/4\pi\bar{r}\right)} = -2P'/\left(\rho + P\right), \quad P' = -\frac{\kappa M\left(\rho + P\right)\left(1 + 4\pi\bar{r}^3 P/M\right)}{8\pi\bar{r}^2\left(1 - \kappa M/4\pi\bar{r}\right)}, \tag{1.27}$$

where the second expression is the Tolman–Oppenheimer–Volkoff equation which is used to construct solutions for relativistic stars. The first equation in (1.26) can be written as $\left[\bar{r}\left(1 - e^{-B}\right)\right]' = \kappa\bar{r}^2\rho$. The only integral that guarantees a regular metric at the origin is $e^{-B} = 1 - \kappa M/4\pi\bar{r}$, where we have introduced the first mass function

$$M \equiv \int_0^{\bar{r}} \mathrm{d}\tilde{\bar{r}}\, 4\pi\tilde{\bar{r}}^2 \rho. \tag{1.28}$$

As is pointed out in [98], the mass defined by (1.28) does not correspond to any invariant quantity whatever, and serves a convenient 'book-keeping' purpose in Schwarzschild-like coordinates. At the *surface* of the fluid $\bar{r} = \bar{R}$, where we are obliged to glue e^B to the Schwarzschild volume element, we find $M_{\mathrm{T}} \equiv M(\bar{R})$, the 'total gravitational mass'. We established already in the introduction that M_{T} is a good physical quantity in these systems, and may be recovered through the methods of Komar and Bondi or Arnowitt, Deser and Misner. The second mass we will consider has a clearer physical motivation at arbitrary radius. It is the integral of the matter density over the proper volume

$$\mathscr{M} \equiv \int_0^{\bar{r}} \mathrm{d}\tilde{\bar{r}}\, 4\pi\tilde{\bar{r}}^2 \rho e^{B/2}. \tag{1.29}$$

The quantity $\mathscr{M}_{\mathrm{T}} \equiv \mathscr{M}(\bar{R})$ is known as the *proper mass* of the fluid. The proper and gravitational masses are related through a quantity $\mathscr{M}_{\mathrm{B}}$ which is traditionally taken to be the gravitational binding energy

$$\mathscr{M}_{\mathrm{T}} = M_{\mathrm{T}} + \mathscr{M}_{\mathrm{B}}. \tag{1.30}$$

We now use the line element (1.23) to introduce a further mass function

$$\mathcal{M} \equiv \int_0^{\bar{r}} \mathrm{d}\tilde{\bar{r}}\, 4\pi\tilde{\bar{r}}^2 \rho e^{A/2+B/2}, \tag{1.31}$$

which, up to a normalisation of $K^{\mathfrak{a}}$, corresponds to the conserved charge mentioned in (1.2). In some sense $\mathcal{M}$ is 'complementary' to $\mathscr{M}$, in that it allows us to define an alternative binding energy

$$\mathcal{M}_{\mathrm{T}} = M_{\mathrm{T}} - \mathcal{M}_{\mathrm{B}}\,. \tag{1.32}$$

The choice of signs in (1.30) and (1.32) reflects the fact that binding energy so defined should be positive in order for the star to be stable: the behaviour of the mass functions $\mathscr{M}_{\mathrm{T}}$ and $\mathcal{M}_{\mathrm{T}}$ is compared in Fig. 1.2 for the Schwarzschild star of gravitational mass M_{T} at various degrees of gravitational collapse. Having introduced $\mathcal{M}$, we notice that the line element (1.23) suggests a second quantity which integrates to M_{T}. The first step is to expand (1.31) by parts

$$\mathcal{M}_{\mathrm{T}} = M_{\mathrm{T}} - \int_0^{\bar{R}} \mathrm{d}\tilde{\bar{r}}\, M\,(A' + B')\, e^{A/2+B/2}/2. \tag{1.33}$$

Then, we can apply relations (1.27) to show $\int_0^{\bar{R}} \mathrm{d}\tilde{\bar{r}} 12\pi\tilde{\bar{r}}^2 P e^{A/2+B/2} = \int_0^{\bar{R}} \mathrm{d}\tilde{\bar{r}} 4\pi\tilde{\bar{r}}^3(\rho A' - PB')e^{A/2+B/2}/2$. From here, by inserting the two field equations of (1.26) and comparing with (1.33) we see that if a mass function is defined by

$$\mathfrak{M} \equiv \int_0^{\bar{r}} \mathrm{d}\tilde{\bar{r}} 4\pi\tilde{\bar{r}}(\rho + 3P)e^{A/2+B/2}, \tag{1.34}$$

we will have agreement with the gravitational mass at the stellar surface $\mathfrak{M}_{\mathrm{T}} = M_{\mathrm{T}}$. The formula (1.34) for $\mathfrak{M}_{\mathrm{T}}$ corresponds to a very powerful definition of gravitational mass in stationary, asymptotically flat spacetimes known as the *Komar mass*. We will briefly outline the physical motivation behind this quantity (see limitations noted in [99]), since the techniques used in the derivation will later need to be imported into the gauge theory of Section 1.3. If a unit test mass is suspended above the star, so that it has four-velocity $u^{\mathfrak{a}} = K^{\mathfrak{a}}/K$, the force applied at spatial infinity to keep it there is $F_{\mathfrak{a}} = u^{\mathfrak{b}}\nabla_{\mathfrak{b}}K_{\mathfrak{a}}$. If such a mass is distributed over closed 2-surface ∂V which contains the star, the observer at infinity must apply an outward force

$$F = \oint_{\partial V} \epsilon_{\mathfrak{c}\mathfrak{d}} n^{\mathfrak{a}} u^{\mathfrak{b}} \nabla_{\mathfrak{b}} K_{\mathfrak{a}}, \tag{1.35}$$

where $n^{\mathfrak{a}}$ is the unit normal to ∂V and $\epsilon_{\mathfrak{c}\mathfrak{d}}$ is the natural volume two-form on ∂V. The static condition ensures $n^{\mathfrak{a}}u_{\mathfrak{a}} \equiv 0$. Now as ∂V is retracted to the asymptotically flat region at spatial infinity, this force may be unambiguously equated with the gravitational mass, and accordingly the Komar mass is defined

$2F = \kappa\mathfrak{M}_{\mathrm{T}}$. An application of the Killing equation to (1.35) allows the Komar mass associated with V to be written in the language of differential forms $\mathfrak{M} \equiv -\kappa^{-1} \oint_{\partial V} \epsilon_{\mathfrak{abcd}} \nabla^{\mathfrak{c}} K^{\mathfrak{d}} \equiv \oint_{\partial V} \alpha_{\mathfrak{ab}}$, where $\epsilon_{\mathfrak{abcd}}$ is the natural volume element on $\mathcal{M}$ imposed by $g_{\mathfrak{ab}}$ and $\alpha_{\mathfrak{ab}}$ is the resultant two-form to be integrated over ∂V. Use of the metric to define volume elements carries an advantage when applying Stokes' theorem $\oint_{\partial V} \alpha_{\mathfrak{ab}} = \int_V d\alpha_{\mathfrak{abc}}$. In the absence of torsion, the operation d which generates an $n+1$-form from an n-form is independent of the derivative operator used to perform it, allowing for the natural choice, $\nabla_{\mathfrak{a}}$. Since the covariant derivative of the natural volume element vanishes, Stokes' theorem can be used to write the mass as a volume integral over the second covariant derivative of $K^{\mathfrak{a}}$ – in this form Stokes' theorem is known as *Gauss' theorem*. The relation for Killing vectors[14] $\nabla_{\mathfrak{a}}\nabla^{\mathfrak{a}}K^{\mathfrak{b}} = -R^{\mathfrak{b}}_{\mathfrak{a}}K^{\mathfrak{a}}$ then enables the Komar mass to be written

$$\mathfrak{M}_{\mathrm{T}} = \frac{2}{\kappa}\int_V \epsilon_{\mathfrak{cde}} R_{\mathfrak{ab}} u^{\mathfrak{a}} K^{\mathfrak{b}}, \tag{1.36}$$

where $\epsilon_{\mathfrak{cde}}$ is the natural volume three-form on V. By exposing the Ricci tensor in the integrand, we see that the contribution from the vacuum vanishes, and so the two-surface may be arbitrarily deformed around the star it encloses. To arrive at (1.36), the Ricci tensor (i.e. a gravitational quantity) was extracted through an integral theorem, due to the ultimate connection between geometry and gravity in GR. We will later see how to reproduce the Komar mass using the gauge theory approach which has no such connection. By applying the Einstein equations to the perfect fluid in (1.25), we see how (1.36) is equivalent to the formula given in (1.34).

In the Newtonian limit we can expect $\bar{r}/\kappa M$ always to be large within the star, and to be of the same order as ρ/P and the Newtonian parameter $\lambda^{-1} \equiv 8\pi\bar{R}/\kappa M_{\mathrm{T}}$. In the same limit, the gravitational force which binds the perfect fluid is expected to follow an r^{-1} potential. Expanding the total proper mass according to $\mathscr{M}_{\mathrm{T}} \equiv \sum_{n=1}^{\infty} \mathscr{M}_{\mathrm{T}n}\lambda^n$, and substituting with (1.27) we find

$$\mathscr{M}_{\mathrm{T}} = M_{\mathrm{T}} + \frac{\kappa}{2}\int_0^{\bar{R}} \mathrm{d}\tilde{\bar{r}}\,\tilde{\bar{r}} M\rho + \mathcal{O}(\lambda^3) = M_{\mathrm{T}} - \int_0^{\bar{R}} \mathrm{d}\tilde{\bar{r}}\, 4\pi\tilde{\bar{r}}^3 P' + \mathcal{O}(\lambda^3). \tag{1.37}$$

If we consider the star to be made up of an ideal gas, a final application of integration by parts shows that the quantity $\mathscr{M}_{\mathrm{B}}$ is equivalent in the Newtonian limit to twice the internal kinetic energy

$$\mathscr{M}_{\mathrm{T}} = M_{\mathrm{T}} + \int_0^{\bar{R}} \mathrm{d}\tilde{\bar{r}}\, 12\pi\tilde{\bar{r}}^2 P + \mathcal{O}(\lambda^3), \tag{1.38}$$

which is a statement of the virial theorem.

Conversely, it is possible to do exactly the same thing with the quantity defined by (1.31). $\mathcal{M}_{\mathrm{T}}$ can be related to M_{T} by an application of integration by parts. Using (1.27) this produces

$$\mathcal{M}_{\mathrm{T}} = M_{\mathrm{T}} - \int_0^{\bar{R}} \mathrm{d}\tilde{\bar{r}}\, M\left(A' + B'\right) e^{A/2+B/2}/2 = M_{\mathrm{T}} + \kappa\int_0^{\bar{R}} \mathrm{d}\tilde{\bar{r}}\,\tilde{\bar{r}} M P' e^{A/2+3B/2}/A'. \tag{1.39}$$

If we now expand (1.39) in the Newtonian limit we will find $e^{A/2+3B/2}/A' = 4\pi\bar{r}^2/\kappa M + \mathcal{O}(\lambda)$. Then as before, a second application of integration by parts gives a complementary statement of the virial

[14] We obtain this using gauge theory methods in Appendix A.5.

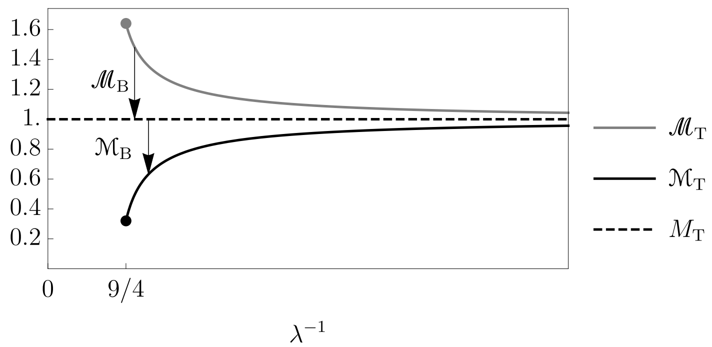


Fig. 1.2 Relativistic binding energies of the Schwarzschild star. The proper mass $\mathscr{M}_{\mathrm{T}}$ is greater than the gravitational mass M_{T}, which is greater than the conserved mass $\mathcal{M}_{\mathrm{T}}$. The relativistic masses diverge as the Newtonian parameter $\lambda^{-1} \equiv 8\pi\bar{R}/\kappa M_{\mathrm{T}}$ shrinks. The Schwarzschild star is unstable for $\lambda^{-1} \leq 9/4$ in Schwarzschild coordinates – though according to Buchdahl's theorem [98] such stars are the most compact that can form.

theorem

$$\mathcal{M}_{\mathrm{T}} = M_{\mathrm{T}} - \int_0^{\tilde{\bar{R}}} \mathrm{d}\tilde{\bar{r}}\, 12\pi\tilde{\bar{r}}^2 P + \mathcal{O}(\lambda^3). \tag{1.40}$$

Using the factor of $\rho + 3P$ in the integrand of (1.34), we can write an alternative to (1.40) which expresses the virial theorem as a *local* property of the perfect fluid, i.e. true at all radii

$$\mathcal{M} = \mathfrak{M} - \int_0^{\tilde{\bar{r}}} \mathrm{d}\tilde{\bar{r}}\, 12\pi\tilde{\bar{r}}^2 P + \mathcal{O}(\lambda^3). \tag{1.41}$$

Of course, interpretation of (1.40) or (1.41) as statements of the virial theorem rely entirely on the interpretation of $\mathcal{M}_{\mathrm{B}}$ as a binding energy in (1.32). Whilst this interpretation is not immediately suggested by (1.2), it looks less arbitrary when we attempt to localise gravitational energy in GTG.

1.3 The view from gauge theory gravity

Whilst it is quite feasible to treat linear gravity as an algebraic problem without ever leaving the physical spacetime $\mathcal{M}$, the procedure in Section 1.2.1 which introduces a flat background $\check{\mathcal{M}}$ is reminiscent of the gauge theory approach. On the one hand, $\mathcal{M}$ is taken to contain some interesting geometry (or 'gravity') imposed by $g_{\mathfrak{ab}}$. On the other, the geometry of the flat background $\check{\mathcal{M}}$ is entirely trivial, $\check{g}_{\mathfrak{ab}}$ imparting it with nothing more than the Minkowskian signature. In $\check{\mathcal{M}}$ the geometry of $\mathcal{M}$ is represented by a collection of tensor fields which, being small, can be managed by a series expansion. This becomes either impractical or impossible in the general case of strong gravitational fields. In gauge theories of gravity, the gravitational gauge fields do not necessarily dictate the geometry of the manifold which contains them (which can *always* be Minkowski space M_4), nor need they be expressed through a series expansion. In the particular case of *gauge theory gravity* (GTG), the nomenclature is influenced by the use of the STA, and M_4 is often referred to as the vector space $\{x\}$. Many previous articles on GTG have included their own substantial primers on the STA, but apart from the new materials provided in Appendices A.1 and A.2 we outline here only the bare principles essential to the gauge theory approach.

The STA is a graded algebra of multivectors spanned by one scalar, four vectors, six bivectors, four trivectors and one pseudoscalar. Particular grades of a multivector are extracted with subscripted chevrons, with the absence of a subscript indicating the scalar (grade-0) part. The fundamental operations are addition and the geometric product, which is denoted by a simple juxtaposition of variables. Particularly useful compositions of these operations are the interior, exterior, commutative and scalar products, respectively $\cdot$, $\wedge$, $\times$ and $*$. We may chose to work either with the coordinate basis $\{\mathsf{e}_\mu\}$ and dual $\{\mathsf{e}^\mu\}$, orthonormal Lorentz-basis $\{\gamma_i\}$ and dual $\{\gamma^i\}$, or with arbitrary constant vectors denoted by lowercase Latin letters such as a and its dual ∂_a. Tensors of second rank are represented by vector-valued linear functions of such vectors, for example the Ricci tensor $\mathcal{R}(a)$, or translational gauge field $\underline{\mathsf{h}}(a)$, which has the same essential function as the tetrad. Whenever a linear function has the same grade as its argument, the underbar-overbar notation is useful in distinguishing the function from its *adjoint*, which roughly corresponds to the commutation of tensor indices. We can also form the inverse of a linear function, for example $\underline{\mathsf{h}}^{-1}(\underline{\mathsf{h}}(a)) \equiv a$. The spin connection is a tensor of third rank, and correspondingly the rotational gauge field is a bivector-valued linear function $\Omega(a) \equiv \omega(\underline{\mathsf{h}}^{-1}(a))$. The derivative with respect to position in $\{x\}$ is simply ∇, whilst the covariant derivative is $\mathcal{D}$. Overdot notation and arrows may be used to indicate the intended target of a derivative operation, replacing nested parentheses.

For a working understanding of these techniques, we recommend either Part I of [69] or Chapter 13 of [93] as compact introductions to GTG. In addition, reference to Chapter 1 of [100], which presents many essential geometric algebra identities in short order, may be very beneficial. An alternative introduction to GTG is to be found in [94], although it differs from our treatment in its emphasis on *gravity frames* $g_\mu \equiv \underline{\mathsf{h}}^{-1}(\mathsf{e}_\mu)$ and $g^\mu \equiv \bar{\mathsf{h}}(\mathsf{e}^\mu)$. Whilst such an approach is more similar to differential geometry, we will instead try to take full advantage of the STA by expressing relations in *frame-free form* wherever possible. We also provide in Appendices A.1 and A.2 a short introduction to the geometric algebra formulation of the general PGT. We will not introduce the tensor fomulation of PGT until Chapter 2, but these appendices can be applied equally to GTG.

1.3.1 The Einstein–Hilbert Lagrangian

The total action of GTG as defined in [69] corresponds to that of Einstein and Hilbert in (1), though we remove the cosmological constant

$$L_{\mathrm{T}} = -\frac{1}{2\kappa}\mathcal{R} + L_{\mathrm{M}}, \quad \mathscr{L}_{\mathrm{T}} \equiv \det \mathsf{h}^{-1} L_{\mathrm{T}}, \quad S_{\mathrm{T}} \equiv \int |\mathrm{d}^4 x| \mathscr{L}_{\mathrm{T}} . \tag{1.42}$$

We can consider gravitational and matter Lagrangian densities as scalar densities on M_4, for these we use a distinct script. This should not be confused with the convention in GTG of using calligraphic script for quantities which are position gauge covariant. The gravitational Lagrangian density $\mathscr{L}_{\mathrm{G}} \equiv L_{\mathrm{G}} \det \mathsf{h}^{-1}$ is therefore the density of the Ricci scalar $\mathcal{R} \equiv \partial_a \cdot \mathcal{R}(a)$, where the Ricci and Riemann tensors are related by $\mathcal{R}(a) \equiv \partial_b \cdot \mathcal{R}(b \wedge a)$,

$$\mathscr{L}_{\mathrm{G}} = -\frac{1}{2\kappa}\mathcal{R}\det \mathsf{h}^{-1} = \frac{1}{\kappa}\left(\partial_a \wedge \partial_b\right) \cdot \Big[\underline{\mathsf{h}}(a) \cdot \dot{\nabla}\dot{\Omega}(\underline{\mathsf{h}}(b)) + \frac{1}{2}\Omega(\underline{\mathsf{h}}(a)) \times \Omega(\underline{\mathsf{h}}(b))\Big]\det \mathsf{h}^{-1}. \tag{1.43}$$

As with ECT, GTG has two equations of motion (E.o.M). Variation with respect to the $\bar{\mathsf{h}}(a)$ and $\Omega(a)$ fields produce the Einstein–Cartan and spin-torsion equations in terms of the *functional* or *dynamical* stress-energy and spin tensors of matter[15], where $\mathcal{G}(a) \equiv \mathcal{R}(a) - \frac{1}{2}a\mathcal{R}$

$$\mathcal{G}(a) = \kappa\tau(a), \quad \tau(\underline{\mathsf{h}}^{-1}(a)) \equiv \det \mathsf{h}\, \partial_{\bar{\mathsf{h}}(a)}(L_{\mathrm{M}}\det \mathsf{h}^{-1}), \quad \mathcal{T}(a) = \kappa\sigma(a), \quad \sigma(\bar{\mathsf{h}}(a)) \equiv \partial_{\Omega(a)}L_{\mathrm{M}}. \tag{1.44}$$

The gauge theory corresponding to GR then follows from a matter Lagrangian in which the rotation gauge fields do not appear, i.e. $L_{\mathrm{M}} = L_{\mathrm{M}}(\Phi|\mathsf{h}, \Omega)$ where the Φ are bosonic matter fields. In this case, the vanishing of the torsion bivector $\mathcal{T}(a)$ allows us to formulate the rotational gauge field in terms of the displacement gauge field

$$\omega(b) = \bar{\mathsf{h}}(\dot{\nabla}) \wedge \dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(b) - \frac{1}{2}b \cdot \left[\partial_c \wedge \bar{\mathsf{h}}(\dot{\nabla}) \wedge \dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(c)\right], \tag{1.45}$$

where $\omega(a) = \Omega(\underline{\mathsf{h}}(a))$ is covariant under displacements, and in practice the following contraction will also be useful

$$\partial_b \cdot \omega(b) = \dot{\bar{\mathsf{h}}}(\dot{\nabla}) - \bar{\mathsf{h}}(\dot{\nabla})\partial_c \cdot \dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(c). \tag{1.46}$$

The invariance of the total action (1.42) under global spacetime translations along some constant vector n allows us to form the *canonical* SET associated with that action

$$\nabla \cdot \underline{\mathsf{t}}(n) = \nabla \cdot (\underline{\mathsf{t}}_{\mathrm{G}}(n) + \underline{\mathsf{t}}_{\mathrm{M}}(n)) = 0, \tag{1.47}$$

where the formulae

$$\underline{\mathsf{t}}_{\mathrm{M}}(n) \equiv \partial_b \langle \phi_{i,n}\partial_{\phi_{i,b}}\mathscr{L}_{\mathrm{M}}\rangle - n\mathscr{L}_{\mathrm{M}}, \quad \underline{\mathsf{t}}_{\mathrm{G}}(n) \equiv \partial_b \langle \Omega(\partial_a)_{,n}\partial_{\Omega(a)_{,b}}\mathscr{L}_{\mathrm{G}}\rangle - n\mathscr{L}_{\mathrm{G}}, \tag{1.48}$$

are adapted from an early exposition on Lagrangian field theories using the STA [101]. In particular, the linear functions in (1.48) are the adjoints of those in [101] so as to agree with conventions regarding the SET of the Dirac field in [93]. Furthermore, following the conventions of [69], *multivector* derivatives with respect to *vector* derivatives of dynamical fields are difficult to work with, therefore the vector derivatives are turned into directional derivatives by contracting with arbitrary constant basis vectors. It should be emphasised that the term *tensor* is used here in a loose sense. Whilst the notion of a tensor is perfectly well defined in the STA, the quantities obtained from (1.48) are in no way constrained to be covariant. A non-covariant linear function is interpreted as a pseudotensor.

Now it is anticipated in [69] and [93] that in the case of Einstein–Hilbert GTG, (1.47) does not yield any new information. If we explore this, we find that whilst the resultant conservation law is indeed a recycling of the field equations, it suggests a curious class of identically conserved currents in the theory. For the gravitational sector, we can substitute (1.43) to give

$$\kappa\underline{\mathsf{t}}_{\mathrm{G}}(n) = \underline{\mathsf{h}}(\partial_a \cdot (n \cdot \dot{\nabla}\dot{\Omega}(\underline{\mathsf{h}}(a)))\det \mathsf{h}^{-1} + \frac{1}{2}n\mathcal{R}\det \mathsf{h}^{-1}. \tag{1.49}$$

[15] The functional SET and spin tensor in (1.43) are related to the geometric algebra definition in Eqs. (A.30) and (A.33) of the SET and spin tensor *densities* employed throughout Chapters 2 to 5 by $\tau(a) \equiv \det \mathsf{h}\underline{\mathsf{h}}^{-1}(\bar{\tau}(a))$ and $\sigma(a) \equiv \det \mathsf{h}\sigma(\bar{\mathsf{h}}^{-1}(a))$.

In the absence of torsion we can find a corresponding *covariantly* conserved current, because $\nabla \cdot J = 0$ implies $\mathcal{D} \cdot \mathcal{J} = 0$, where

$$\mathcal{J} \equiv \underline{\mathsf{h}}^{-1}(J)\det \mathsf{h}. \tag{1.50}$$

For matter without a spin tensor, the covariantised quantity is the functional SET[16]

$$\underline{\mathsf{h}}^{-1}(\underline{\mathsf{t}}_{\mathrm{M}}(n))\det \mathsf{h} = \kappa\tau(\underline{\mathsf{h}}^{-1}(n)), \quad \underline{\mathsf{h}}^{-1}(\underline{\mathsf{t}}_{\mathrm{G}}(n))\det \mathsf{h} = -\mathcal{R}(\underline{\mathsf{h}}^{-1}(n)) + \frac{1}{2}\underline{\mathsf{h}}^{-1}(n)\mathcal{R} + \mathcal{D}\cdot\omega(\underline{\mathsf{h}}^{-1}(n)). \tag{1.51}$$

Assembling these results, we see that global spacetime translations give rise to the conservation law

$$\mathcal{D}\cdot[-\mathcal{G}(\underline{\mathsf{h}}^{-1}(n)) + \kappa\tau(\underline{\mathsf{h}}^{-1}(n)) + \mathcal{D}\cdot\Omega(n)] = 0. \tag{1.52}$$

As expected, this law does not tell us anything new. The first two terms in brackets can immediately be removed using the Einstein equation (1.44). Furthermore, the final term can also be removed using a law which we find applies to general bivectors B in the absence of torsion

$$\mathcal{D}\cdot(\mathcal{D}\cdot B) = 0. \tag{1.53}$$

To prove this new law, let us write

$$\mathcal{D}\cdot(\mathcal{D}\cdot B) = (\mathcal{D}\wedge\bar{\mathsf{h}}(\partial_b))\cdot\mathcal{D}_b B + \bar{\mathsf{h}}(\partial_a\wedge\partial_b)\cdot\overrightarrow{\mathcal{D}}_a\mathcal{D}_b B = (\partial_a\wedge\partial_b)\cdot(\mathcal{R}(a\wedge b)\times B), \tag{1.54}$$

where we have assumed a to be an arbitrary constant. Some further identities give

$$(\partial_a\wedge\partial_b)\cdot(\mathcal{R}(a\wedge b)\times B) = \mathcal{R}(a\wedge b)\cdot(B\times(\partial_a\wedge\partial_b)) = -2(\partial_a\wedge\mathcal{R}(a))\cdot B = 0, \tag{1.55}$$

and the last equality is a result of the symmetry of the Ricci tensor. Note that covariance of B is not required [17]. Equation (1.53) provides us with an instant formula for generating conserved vector currents in GTG. Given covariant vectors $\mathcal{U}$ and $\mathcal{V}$ we have

$$\mathcal{J} = \mathcal{D}\cdot(\mathcal{U}\wedge\mathcal{V}). \tag{1.56}$$

In fact, only one vector field is necessary: setting $\mathcal{U} = \mathcal{D}$ allows us to construct the conserved currents introduced by Komar in [76]

$$\mathcal{J} = \mathcal{D}\cdot(\mathcal{D}\wedge\mathcal{V}) - \dot{\mathcal{D}}\cdot(\dot{\mathcal{D}}\wedge\mathcal{V}). \tag{1.57}$$

It is clear from (1.57) that from the gauge theory perspective, Komar currents are a composite of identically conserved currents and consequently, as a whole, are purely second-order in the covariant derivatives of $\mathcal{V}$. The useful application of the currents corresponding to the first term in (1.57) is left for further work.

[16]In the presence of spin, the $\Omega(a)$ dependence in the Lagrangian significantly complicates the picture.
[17]We can, in fact, arrive at (1.53) through the powerful 'double wedge' relation (see Appendix A.5) for arbitrary multivector, M, in the absence of torsion $\mathcal{D}\wedge(\mathcal{D}\wedge M) = 0$. To see this we set $M = IB$ and use the pseudoscalar to convert between interior and exterior products $\mathcal{D}\wedge(\mathcal{D}\wedge IB) = I\mathcal{D}\cdot(\mathcal{D}\cdot B)$.

Now the observations that have been made about the covariantised equation (1.52), could in principle be made just as well in the flat space as follows. By writing B as an exterior product of two arbitrary vectors (or a sum thereof), we can show that $\mathcal{D}\cdot B = \nabla\cdot(\underline{\mathsf{h}}(B)\det\mathsf{h}^{-1})$, and in this way the term in question is revealed to be the identically conserved gradient of a *superpotential* buried in the Einstein tensor $\nabla\cdot(\nabla\cdot\underline{\mathsf{h}}(\omega(\underline{\mathsf{h}}^{-1}(n)))\det\mathsf{h}^{-1}) = 0$. As mentioned in Section 1.1, the identification of superpotentials which split the Einstein tensor has been a fruitful approach to finding gravitational SETs and pseudotensors. Thus we see that the variational approach to gravitational stress-energy localisation in Einstein–Hilbert GTG introduces its own split in (1.52). The gravitational SET (or pseudotensor) implied by (1.52) contains gradients of the rotational gauge field, which, once substituted for by (1.45) will become second derivatives of the displacement gauge field. In the next section we will see how this can be avoided by an alternative choice of gravitational Lagrangian.

1.3.2 Møller's pseudotensor

The Lagrangian (1.42) has the advantages of simplicity and covariance, but neither of these properties is necessary to reproduce the field equations. In obtaining the complex (1.5), Einstein removed any second derivatives of the metric appearing in the Einstein–Hilbert Lagrangian by means of a surface term, since a Lagrangian which is homogeneously second-order in ∂g must produce a canonical stress energy (affine) tensor with that same property. Similarly, one could use a surface term to lever the vector derivative off the rotation gauge fields and onto the displacement gauge fields in (1.42), and hope that the resulting pseudotensor will have more desirable properties than that obtained above. The new Lagrangian is ${}_{\mathrm{M}}\mathscr{L}_{\mathrm{G}}(\bar{\mathsf{h}}(a),\bar{\mathsf{h}}(a)_{,b},\Omega(a)) \equiv \mathscr{L}_{\mathrm{G}}(\bar{\mathsf{h}}(a),\Omega(a),\Omega(a)_{,b}) - \nabla\cdot{}_{\mathrm{M}}\mathscr{F}$, and by inspection of (1.43), the 'minimal' choice of surface term is simply $\kappa_{\mathrm{M}}\mathscr{F} \equiv \underline{\mathsf{h}}(\partial_a\cdot\omega(a))\det\mathsf{h}^{-1}$, with the new gravitational Lagrangian given by the formula

$$\begin{aligned}\kappa_{\mathrm{M}}\mathscr{L}_{\mathrm{G}} \equiv (\partial_a\wedge\partial_b)\cdot\Big[&\Omega(\underline{\mathsf{h}}(b))\nabla\cdot\big(\det\mathsf{h}^{-1}\underline{\mathsf{h}}(a)\big) + \det\mathsf{h}^{-1}\underline{\mathsf{h}}(a)\cdot\dot{\nabla}\Omega(\dot{\underline{\mathsf{h}}}(b)) \\ &+ \frac{1}{2}\Omega(\underline{\mathsf{h}}(a))\times\Omega(\underline{\mathsf{h}}(b))\det\mathsf{h}^{-1}\Big].\end{aligned} \tag{1.58}$$

The subscript indicates that this 'minimal' modification to the gravitational action is analogous to the effective Lagrangian of Møller. This is not surprising, given that Møller was working at the level of the tetrad. By applying (1.45) and (1.46) we find that (1.58) can be written in the very compact form

$$\kappa_{\mathrm{M}}\mathscr{L}_{\mathrm{G}} \equiv -\frac{1}{2}(\partial_a\wedge\partial_b)\cdot(\omega(a)\times\omega(b))\det\mathsf{h}^{-1}. \tag{1.59}$$

Since ${}_{\mathrm{M}}\mathscr{L}_{\mathrm{G}}\det\mathsf{h}$ is dependent only on the $\omega(a)$ fields, we see that it is still translation gauge covariant, having lost only the rotational gauge invariance of $\mathcal{R}$. The symmetry of the new action under global spacetime translations again implies a conservation law on the flat background. This time it is convenient to evaluate the adjoint form corresponding to (1.48)

$${}_{\mathrm{M}}\bar{\mathfrak{t}}_{\mathrm{G}}(n) \equiv \dot{\nabla}\langle\dot{\bar{\mathsf{h}}}(\partial_a)\partial_{\bar{\mathsf{h}}(a),n}{}_{\mathrm{M}}\mathscr{L}_{\mathrm{G}}\rangle - n\,{}_{\mathrm{M}}\mathscr{L}_{\mathrm{G}}\,. \tag{1.60}$$

Though less transparent than (1.59), the expanded form (1.58) is easier to work with. To find the contribution of the first term in (1.58) to that in (1.60) we will need

$$\dot{\nabla}\langle(\partial_a \wedge \partial_b)\cdot\Omega(\underline{\mathsf{h}}(b))\dot{\bar{\mathsf{h}}}(\partial_c)\partial_{\bar{\mathsf{h}}(c),n}(\det\mathsf{h}^{-1}a\cdot\underline{\mathsf{h}}(\partial_d)_{,d}+\partial_d\cdot\underline{\mathsf{h}}(a)(\det\mathsf{h}^{-1})_{,d})\rangle \\ = \dot{\nabla}(\partial_a\wedge\partial_b)\cdot\omega(b)(a\cdot\dot{\bar{\mathsf{h}}}(n)-n\cdot\underline{\mathsf{h}}(a)\partial_c\cdot\dot{\underline{\mathsf{h}}}\underline{\mathsf{h}}^{-1}(c))\det\mathsf{h}^{-1}, \tag{1.61}$$

where we make use of the identity $\partial_{\bar{\mathsf{h}}(c),n}(\det\mathsf{h}^{-1})_{,b} = -(n\cdot b)\det\mathsf{h}^{-1}\underline{\mathsf{h}}^{-1}(c)$ from Appendix A.6. Meanwhile the second term in (1.58) contributes

$$\dot{\nabla}\bar{\mathsf{h}}(\partial_a)\cdot(\partial_b\cdot\Omega(\partial_d\dot{\bar{\mathsf{h}}}(\partial_c)\cdot(\partial_{\bar{\mathsf{h}}(c),n}b\cdot\bar{\mathsf{h}}(d)_{,a})))\det\mathsf{h}^{-1} = \dot{\nabla}(\bar{\mathsf{h}}(n)\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(\partial_b))\cdot\omega(b)\det\mathsf{h}^{-1}. \tag{1.62}$$

If we assemble these we arrive at the following formula for the pseudotensor of Møller expressed as a linear function and in terms of the gravitational gauge fields

$$\kappa_{\mathrm{M}}\bar{\mathsf{t}}_{\mathrm{G}}(n) = \dot{\nabla}(\dot{\bar{\mathsf{h}}}(n)\wedge\partial_b-\bar{\mathsf{h}}(n)\wedge\partial_b\partial_c\cdot\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(c)+\bar{\mathsf{h}}(n)\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(\partial_b))\cdot\omega(b)\det\mathsf{h}^{-1}-\kappa n_{\mathrm{M}}\mathscr{L}_{\mathrm{G}}. \tag{1.63}$$

We will make use of this explicit formula in the final section. Note that the trace of the Møller pseudotensor reduces to the effective Lagrangian $\kappa\partial_n\cdot{}_{\mathrm{M}}\bar{\mathsf{t}}_{\mathrm{G}}(n) = 3(\partial_a\wedge\partial_b)\cdot(\omega(a)\times\omega(b))\det\mathsf{h}^{-1}$, yet the pseudotensor itself cannot be expressed purely in terms of the $\omega(a)$: Møller's superpotential is tensorial, so this cannot also be true of its energy-momentum complex and pseudotensor. An alternative form for the pseudotensor is given in Appendix A.4.

As before, we will expect the conservation law on M_4 to be ${}_{\mathrm{M}}\dot{\bar{\mathsf{t}}}(\dot{\nabla})=0$, where ${}_{\mathrm{M}}\bar{\mathsf{t}}(n)\equiv{}_{\mathrm{M}}\bar{\mathsf{t}}_{\mathrm{G}}(n)+\bar{\mathsf{t}}_{\mathrm{M}}(n)$, and in order to obtain some very useful results we will arrive at this by the same route taken by Dirac when discussing the complex of Einstein in [102]. A very useful consequence of the field equations and the contracted Bianchi identity is the covariant conservation law (1.1) with which we began our discussion

$$\dot{\tau}(\dot{\mathcal{D}}) = 0. \tag{1.64}$$

This can be expanded as a vector derivative with two Levi–Civita connection terms $\dot{\tau}(\dot{\mathcal{D}}) = \dot{\tau}(\bar{\mathsf{h}}(\dot{\nabla}))+\tau(\partial_b\cdot\omega(b))-\tau(\partial_c)\cdot\omega(c)$. Having performed the Palatini variation, we can eliminate the connection in terms of the displacement gauge field in the absence of torsion with (1.45), and use the symmetry of the functional SET of matter $\tau(a)$, to write

$$\dot{\tau}(\bar{\mathsf{h}}(\dot{\nabla}))+\tau(\dot{\bar{\mathsf{h}}}(\dot{\nabla}))-\tau(\bar{\mathsf{h}}(\dot{\nabla}))\partial_b\cdot\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(b)-\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}\tau(\bar{\mathsf{h}}(\dot{\nabla}))+\bar{\mathsf{h}}(\dot{\nabla})\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(\partial_c)\cdot\tau(c)=0. \tag{1.65}$$

Finally, by applying the displacement gauge field we can collect some terms into a convenient *total* divergence on M_4

$$\bar{\mathsf{h}}^{-1}(\dot{\tau}(\dot{\mathcal{D}}))\det\mathsf{h}^{-1} = \bar{\mathsf{h}}^{-1}(\tau(\bar{\mathsf{h}}(\overleftrightarrow{\nabla})))\det\mathsf{h}^{-1}+\dot{\nabla}\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(\partial_c)\cdot\tau(c)\det\mathsf{h}^{-1}=0. \tag{1.66}$$

So long as the matter is not a source of spin, the linear function acted on by this total divergence is seen to be its canonical SET $\bar{\mathsf{t}}_{\mathrm{M}}(a)=\bar{\mathsf{h}}^{-1}\tau(\bar{\mathsf{h}}(a))\det\mathsf{h}^{-1}$. The final term then expresses the exchange of energy-momentum with the gravitational field on M_4

$$_{\mathrm{M}}\dot{\bar{\mathsf{t}}}_{\mathrm{G}}(\dot{\nabla}) = \dot{\nabla}\langle\partial_b\cdot\nabla(\dot{\bar{\mathsf{h}}}(\partial_a)\partial_{\bar{\mathsf{h}}(a),b}{}_{\mathrm{M}}\mathscr{L}_{\mathrm{G}})\rangle - \nabla_{\mathrm{M}}\mathscr{L}_{\mathrm{G}} = \kappa\dot{\nabla}\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(\partial_a)\cdot\mathsf{\tau}(a)\det\mathsf{h}^{-1}, \tag{1.67}$$

where we have assumed for the final equality that, since the Lagrangian has been modified only by a surface term, the field equations (1.44) should be unchanged As a result we find that energy and momentum are conserved on M_4 in the expected manner $_{\mathrm{M}}\dot{\underline{\mathsf{t}}}_{\mathrm{G}}(\dot{\nabla})+\dot{\underline{\mathsf{t}}}_{\mathrm{M}}(\dot{\nabla})=0$, but in the process we have equated the divergence of Møller's pseudotensor on M_4 to the final term in (1.67), and this expression will be useful later.

1.3.3 The general Klein–Gordon correspondence

In Sections 1.2.2 and 1.2.3 we developed a natural way to extend the tensor of Butcher to the pseudotensor of Einstein. Now the correspondence between GTG and ECT has led us to the pseudotensor of Møller rather than that of Einstein, but it is observed in [94] that the two are equivalent in the spacetimes under discussion in this chapter. A first application of this result is that the Klein–Gordon correspondence of Section 1.2.5, displayed by Butcher's tensor in the Newtonian limit, fully survives the 'nonlinearisation' to Møller's pseudotensor in the presence of strong gravitational fields. In this regime the matter density ρ is not expected to fully generate the gravitational potential φ: the whole of the matter SET acts as the source in any theory of gravity and so we anticipate that the pressure P will play a part. Therefore we denote the general source density for φ by ϱ such that $\varrho\equiv\rho+\mathcal{O}(\lambda^2)$, where λ is the Newtonian parameter. We will soon see that this generalisation is useful in the context of the relativistic mass functions discussed in Section 1.2.5.

We first seek to understand what the Klein–Gordon theory on $\check{\mathcal{M}}$ discussed in Section 1.2.4 looks like in the STA on M_4. We stress that this is precisely the same field theory (1.22) on Minkowski spacetime, but expressed using the apparatus of geometric algebra. The Lagrangian (1.22) will be $L_{\mathrm{KG}}\equiv(\nabla\varphi)^2-\kappa\varphi\varrho$. This is a Lagrangian density directly on M_4, and so it is not necessarily gauge covariant. The canonical SET, which is symmetric, can be partitioned into a field term and an interaction term

$$_{\mathrm{KG}}\mathsf{t}(n) = 2\nabla\varphi n\cdot\nabla\varphi - n(\nabla\varphi)^2+\kappa n\varphi\varrho = \nabla\varphi n\nabla\varphi-\kappa n\varphi. \tag{1.68}$$

The E.o.M (1.20) will then be simply $\Box\varphi=-\kappa\varrho/2$. Rectangular isotropic coordinates corresponding to the line element (1.24) are introduced through the displacement gauge fields $\underline{\mathsf{h}}^{-1}(\gamma_0)=e^{A/2}\gamma_0$ and $\underline{\mathsf{h}}^{-1}(\gamma_i)=e^{C/2}\gamma_i$ acting on an orthonormal basis – these are expected to go over to (1.21) in the Newtonian limit. The choice of timelike Killing vector $\mathcal{K}=\underline{\mathsf{h}}^{-1}(\gamma_0)=g_0$ is then made for us. The use of isotropic coordinates enables us to make two simplifications. Firstly the displacement gauge field will be self-adjoint, so we can dispense with over/underbar notation. Secondly, because the spacetime is static, we can assume the action of all vector derivatives is purely spatial. In practice this is a considerable shortcut, for instance the connection is simply $\omega(b)=\Omega(\mathsf{h}(b))=\mathsf{h}(\dot{\nabla})\wedge\dot{\mathsf{h}}\mathsf{h}^{-1}(b)$. Møller's pseudotensor given by the linear function in (1.63) is then seen to be symmetric

$$_{\mathrm{M}}\mathsf{t}_{\mathrm{G}}(n)=\frac{1}{4}e^{(A+C)/4}\big[\big((\dot{A}+2\dot{C})(\ddot{A}+2\ddot{C})-\dot{A}\ddot{A}-2\dot{C}\ddot{C}\big)\ddot{\nabla}\dot{\nabla}\cdot n-(\dot{C}\ddot{C}+2\dot{A}\ddot{C})\dot{\nabla}\cdot\ddot{\nabla}n\big]. \tag{1.69}$$

The picture is further simplified when we assume that the spacetime is equipped with symmetry such that $\nabla A\wedge\nabla C=0$, and this is clearly the case for the spherically symmetric spacetimes. We then see that Møller's pseudotensor does indeed adopt the form of the field part of (1.68) ${}_{\mathrm{M}}\mathsf{t}_{\mathrm{G}}(n) = \nabla\varphi n\nabla\varphi$, with the radial field strength associated with the gravitational scalar potential given by[18]

$$\varphi' = \frac{1}{2}e^{(A+C)/4}\sqrt{C'^2 + 2A'C'}. \tag{1.70}$$

We have uncovered a remarkably compact picture of gravitational energetics: the stress and energy of the gravitational field on M_4 coincides with that of a scalar field φ. It should be stressed that as with the linearised version of this relationship, the link with the Lagrangian L_{KG} is completely formal: if we want to construct an E.o.M we will have to assemble it by hand rather than from an Euler–Lagrange equation. For the spherically symmetric perfect fluid this is

$$\Box\varphi = \frac{\kappa}{8}e^{(A+3C)/2}\left[\rho A' - 3PC'\right]/\varphi'. \tag{1.71}$$

In fact this formula is not unique to the spherical case: it is the isotropic form of the general relation (1.67) which we obtained as part of the conservation law on M_4 for Møller's pseudotensor. An obvious consequence of (1.71) is that φ obeys the Laplace equation in a vacuum. Particularly, for the case of relativistic stars, it is seen that φ in the Schwarzschild spacetime above the stellar surface appears to have been generated by the gravitational mass of the star

$$\varphi = -\kappa M_{\mathrm{T}}/8\pi r. \tag{1.72}$$

Then, we see that not only does the Klein–Gordon correspondence hold in the presence of strong gravitational fields, but the gravitational potential retains its Newtonian form! We are now in a position to equate ϱ with the RHS of (1.71). Doing so, the familiar Newtonian formula (1.72) then indicates that ϱ describes a *gravitational* mass density on M_4: this refinement could not be made in the Newtonian limit where ϱ and ρ were indistinguishable. Finally, if we expand φ in some Newtonian parameter λ so that $\varphi \equiv \sum_{n=1}^{\infty}\varphi_n\lambda^n$ and $\varrho \equiv \sum_{n=1}^{\infty}\varrho_n\lambda^n$, we can write the Newtonian limit of rectangular isotropic coordinates as $e^{A/2} = 1 + 2\varphi_1\lambda + \mathcal{O}(\lambda^2)$ and $e^{C/2} = 1 - 2\varphi_1\lambda + \mathcal{O}(\lambda^2)$, where the Newtonian potential is $\Box\varphi_1\lambda = -\kappa\rho/2$. The Poisson-like equation then expands to give us

$$\varrho_1\lambda + \varrho_2\lambda^2 = \rho(1 - 2\varphi_1\lambda) + 3P = \rho e^{(A+3C)/2} + 3P + \mathcal{O}(\lambda^3), \tag{1.73}$$

which is precisely the local virial theorem we anticipated in (1.41), only it is expressed in isotropic coordinates. To compare, a hypothetical localisation of gravitational mass $\tilde{\varrho}$ designed to reproduce the *conventional* virial theorem, (1.38), would instead obey

$$\tilde{\varrho}_1\lambda + \tilde{\varrho}_2\lambda^2 = \rho(1 - 3\varphi_1\lambda) - 3P = \rho e^{3C/2} - 3P + \mathcal{O}(\lambda^3). \tag{1.74}$$

In this way we connect back to the definitions of relativistic mass discussed in Section 1.2.5.

1.3.4 Mass in gauge theory gravity

We conclude our discussion of the gauge theory approach with an attempt to place the conserved mass $\mathcal{M}_{\mathrm{T}}$ mentioned in (1.2) and (1.32) in the global picture. In Section 1.3.2, we introduced equation (1.64) as

[18] Since we are now using isotropic coordinates, the prime denotes differentiation with respect to r in the line element (1.24), rather than the Schwarzschild radial coordinate $\bar{r}$ used in Section 1.2.5.

a consequence of the contracted Bianchi identity and Einstein equations. For any vector field V we have $\dot{\mathcal{D}}\cdot\dot{\mathcal{T}}(V)=0$, which is the gauge theory statement of the non-conservation of material energy-momentum currents discussed in Section 1.2.1. In particular, if the V were taken to be any of the basis vectors γ_i (or even a physically meaningful vector such as the four-velocity of a local observer u), we can see that (1.8) is equivalent to the result $\mathcal{D}\cdot\tau(V)=\partial_a\cdot\tau(a\cdot\mathcal{D}V)\neq 0$. The matter energy momentum currents *are* conserved, however, if for V we take any of the Killing vectors $\mathcal{K}$, which embody the symmetries of a given spacetime through the equation

$$a\cdot(b\cdot\mathcal{D}\mathcal{K})=-b\cdot(a\cdot\mathcal{D}\mathcal{K}), \tag{1.75}$$

leaving us with a fully covariant and covariantly conserved vector current

$$\mathcal{D}\cdot\tau(\mathcal{K})=0. \tag{1.76}$$

To further develop our picture on M_4, we can reverse the use of (1.50) in Section 1.3.1 to construct a corresponding conserved current

$$\nabla\cdot\underline{\mathsf{h}}(\tau(\mathcal{K}))\det\mathsf{h}^{-1}=0. \tag{1.77}$$

Generally, conserved charges are obtained from conservation laws. A timelike observer in M_4 has proper time τ, which can be used as a coordinate function to define timelike basis vectors e_τ and e^τ. Since e_τ is the four-velocity of the observer, it must be a unit vector. The conserved charge density associated with the current (1.77) over the whole spatial hypersurface Σ_τ can be integrated

$$\mathcal{Q}_{\mathrm{T}}\equiv\int_{\Sigma_\tau}|\mathrm{d}^3x|\mathsf{e}^\tau\cdot\underline{\mathsf{h}}(\tau(\mathcal{K}))\det\mathsf{h}^{-1}=\int_{\Sigma_\tau}\langle\mathcal{P}_\perp(\tau(\mathcal{K}))\underline{\mathsf{h}}^{-1}(\mathrm{d}^3x)\mathring{I}^{-1}\rangle, \tag{1.78}$$

where $\mathcal{P}_\perp(a)$ is the gauge invariant rejection operator and $\mathring{I}$ the pseudoscalar associated with Σ_τ - both are defined in Appendix A.3. We have chosen a calligraphic script for $\mathcal{Q}_{\mathrm{T}}$ because from (1.78) we can write it in covariant form

$$\mathcal{Q}_{\mathrm{T}}\equiv\int_{\Sigma_t}\langle\mathcal{T}(\mathcal{K})\underline{\mathsf{h}}^{-1}(\mathrm{d}^3x)I^{-1}\rangle. \tag{1.79}$$

Now we have $\mathcal{Q}_{\mathrm{T}}$ in gauge-covariant form it is easier to interpret. By applying Gauss' law to (1.79) we see that $\mathcal{Q}_{\mathrm{T}}$ is independent of our choice of Σ_τ because of (1.76). This is equivalent to the observation in M_4 that $\mathcal{Q}_{\mathrm{T}}$ appears as a conserved charge in the theory. It is now clear that we are dealing with the same quantity Q_{T} mentioned in (1.2).

The formula (1.79) is covariant, but still depends on the normalisation of $\mathcal{K}$. The other relativistic mass which used $\mathcal{K}$ was that of Komar $\mathfrak{M}_{\mathrm{T}}$, found to be equal to the gravitational mass M_{T} – for the Komar mass we normalised the time-like Killing vector to unity at spatial infinity in order to take advantage of the Newtonian regime. In gauge theory terms, the force applied by the observer at spatial infinity to suspend a unit mass with four-velocity $u=\mathcal{K}/|\mathcal{K}|$ is $\mathcal{F}=u\cdot\mathcal{D}\mathcal{K}=u\cdot(\mathcal{D}\wedge\mathcal{K})$, where the Killing equation (1.75) can be used to recognise the presence of the bivector. In Section 1.2.5 the Komar integral was performed in the hypersurface orthogonal to $\mathcal{K}$: we can set up a coordinate system that reflects this by taking $\mathcal{K}=\underline{\mathsf{h}}^{-1}\mathsf{e}_t=g_t$ and integrating in the surface Σ_t. By applying Gauss' law (A.13), we can find the Komar mass within some bounded region V in Σ_t as $\mathfrak{M} = \oint_{\partial V} \langle \mathcal{D} \wedge \mathcal{K} \underline{\mathsf{h}}^{-1}(\mathrm{d}^2 x) I^{-1} \rangle = \int_V \langle \mathcal{D} \cdot (\mathcal{D} \wedge \mathcal{K}) \underline{\mathsf{h}}^{-1}(\mathrm{d}^3 x) I^{-1} \rangle$. We can then replace the second covariant derivative of the Killing field $\overleftrightarrow{\mathcal{D}} \cdot \mathcal{D}\mathcal{K} = \partial_a \mathcal{R}(a) \cdot \mathcal{K}$ using (A.20), and then the total Komar mass is found by extending V over the whole spatial hypersurface $\mathfrak{M}_{\mathrm{T}} = \int_{\Sigma_t} \langle \mathcal{R}(\mathcal{K}) \underline{\mathsf{h}}^{-1}(\mathrm{d}^3 x) I^{-1} \rangle$. If we adopt the same normalisation of $\mathcal{K}$ used above in the conserved charge, we recover the mass $\mathcal{Q}_{\mathrm{T}} = \mathcal{M}_{\mathrm{T}}$ mentioned in Section 1.2.5.

We claimed in Section 1.1 that a viable energy-momentum complex ought to integrate to M_{T}. We have also shown that $\mathcal{M}_{\mathrm{T}}$ can be thought of as a conserved charge on M_4, and that the pseudotensor of Møller appears as the SET of a scalar field there. We should therefore conclude by balancing the energy budget directly on M_4 as well, by re-introducing the isotropic coordinates and orthonormal basis vectors. This system of coordinates is an example of the kind we have just been considering. The energy density of the gravitational field on the background is given by ${}_{\mathrm{M}}U_{\mathrm{G}} \equiv \gamma_0 \cdot {}_{\mathrm{M}}\mathsf{t}_{\mathrm{G}}(\gamma_0) = -{}_{\mathrm{M}}\mathscr{L}_{\mathrm{G}}$, so from the definition (1.58) of Møller's effective Lagrangian

$$\int_V |\mathrm{d}^3 x| {}_{\mathrm{M}}U_{\mathrm{G}} = \int_V |\mathrm{d}^3 x| \Big(\frac{1}{2\kappa} \mathcal{R} \det \mathsf{h}^{-1} - \nabla \cdot {}_{\mathrm{M}}\mathscr{F} \Big). \tag{1.80}$$

If we apply the Einstein equations to the Ricci scalar in (1.80) we see $\mathcal{R} = \kappa(3P - \rho)$, so we have

$$\int_V |\mathrm{d}^3 x| ({}_{\mathrm{M}}U_{\mathrm{G}} + \rho \det \mathsf{h}^{-1}) = \frac{1}{2} \int_V |\mathrm{d}^3 x| (\rho + 3P) \det \mathsf{h}^{-1} - \oint_{\partial V} |\mathrm{d}^2 x| \mathsf{e}_r \cdot {}_{\mathrm{M}}\mathscr{F}. \tag{1.81}$$

The first term on the RHS of (1.81) is identifiable as $M_{\mathrm{T}}/2$ since it is equivalent to the Komar mass integral. For asymptotically flat systems, the second term on the RHS approaches $M_{\mathrm{T}}/2$ when evaluated at spatial infinity, so we have

$$\int_{\Sigma_t} |\mathrm{d}^3 x| ({}_{\mathrm{M}}U_{\mathrm{G}} + \rho \det \mathsf{h}^{-1}) = M_{\mathrm{T}}. \tag{1.82}$$

The second term on the LHS of (1.82) will integrate to $\mathcal{M}_{\mathrm{T}}$, again by comparison with Section 1.2.5. We can then interpret Møller and Einstein's account of the energy sequestered in the gravitational field as the binding energy of the system, allowing us to justify (1.32), the energy relation

$$\mathcal{M}_{\mathrm{B}} + \mathcal{M}_{\mathrm{T}} = M_{\mathrm{T}}. \tag{1.83}$$

1.3.5 Example: Schwarzschild star

It is constructive to illustrate the picture of gravitostatic energetics we have been building using a simple system. The simplest static spherically symmetric perfect fluid is the *Schwarzschild star*, which has a constant proper density $\rho = \rho_0$ and pressureless surface at $\bar{r} = \bar{R}$ in Schwarzschild-like coordinates. Below the stellar surface, the functions appearing in the Schwarzschild line element (1.23) are

$$e^{A/2} = \frac{1}{2} \Big(3\sqrt{1 - \kappa M_{\mathrm{T}}/4\pi\bar{R}} - \sqrt{1 - \kappa M/4\pi\bar{r}} \Big), \quad e^{B/2} = 1/\sqrt{1 - \kappa M/4\pi\bar{r}}. \tag{1.84}$$

In terms of the Newtonian parameter $\lambda^{-1} \equiv 8\pi\bar{R}/\kappa M_{\mathrm{T}}$, the star has proper mass

$$\mathscr{M}_{\mathrm{T}} = \frac{3}{8}\sqrt{\lambda^{-1}} M_{\mathrm{T}} \Big[-2\sqrt{\lambda^{-1} - 2} + \sqrt{2}\lambda^{-1} \tan^{-1} \Big(\sqrt{2/(\lambda^{-1} - 2)} \Big) \Big], \tag{1.85}$$

and conserved mass

$$\mathcal{M}_{\mathrm{T}} = \frac{1}{16} M_{\mathrm{T}} \Big[-18\lambda^{-1} + 28 + 9\sqrt{2}\lambda^{-1}\sqrt{\lambda^{-1}-2}\tan^{-1}\left(\sqrt{2/(\lambda^{-1}-2)}\right)\Big]. \tag{1.86}$$

In these formulae, the Schwarzschild coordinate mass function is simply $M = M_{\mathrm{T}}\bar{r}^3/\bar{R}^3$. Superficially the functions (1.85) and (1.86) appear similar, and indeed both agree on the Newtonian limit of the binding energy $\mathscr{M}_{\mathrm{B}} = \mathcal{M}_{\mathrm{B}} = 3M_{\mathrm{T}}/5\lambda^{-1} + \mathcal{O}(\lambda^2)$. As is shown in Fig. 1.2 however, $\mathscr{M}_{\mathrm{T}} > M_{\mathrm{T}} > \mathcal{M}_{\mathrm{T}}$, so we see that in the alternative interpretation (1.32), the gravitational mass loses a positive binding energy.

To see the Klein–Gordon correspondence in action, we base our displacement gauge fields beneath the stellar surface on the isotropic coordinates first set out by Wyman [103]. The functions appearing in the line element (1.24) are

$$e^{A/2} = \frac{16\pi R - 2\kappa M_{\mathrm{T}} + \kappa M_{\mathrm{T}}(32\pi R - \kappa M_{\mathrm{T}})r^2/16\pi R^3}{(16\pi R + \kappa M_{\mathrm{T}})(1+\kappa M_{\mathrm{T}} r^2/16\pi R^3)}, \quad e^{C/2} = \frac{(1+\kappa M_{\mathrm{T}}/16\pi R)^3}{1+\kappa M_{\mathrm{T}} r^2/16\pi R^3}. \tag{1.87}$$

By substituting (1.87) into (1.70) we find the radial 'gravitational field strength' beneath the stellar surface to be

$$\varphi' = \frac{\kappa M_{\mathrm{T}}\left(1+\frac{\kappa M_{\mathrm{T}}}{16\pi R}\right) r\sqrt{1+\frac{\kappa M_{\mathrm{T}}}{16\pi R} - \frac{\kappa M_{\mathrm{T}} r^2}{16\pi R^3}\left(1-\frac{\kappa M_{\mathrm{T}}}{32\pi R}\right)}}{8\pi R^3\left(1+\kappa M_{\mathrm{T}} r^2/16\pi R^3\right)^2}. \tag{1.88}$$

In Fig. 1.1 we show a pair of Schwarzschild stars with the same gravitational mass M_{T}, but which have stalled their collapse at different isotropic radii. The integrated gravitational potential φ takes the same Newtonian form (1.72) above the surface of each star. The source density ϱ does not share the uniform distribution of the proper mass: we see that the localisation of gravitational matter it represents tends to accumulate at the stellar core.

1.4 Closing remarks

It is difficult to gather from [90, 87–89] a single *motivating* definition of ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$ which invites generalisation to nonlinear gravity. The stress-energy and spin tensors emerge as satisfying a series of physically motivated requirements, any of which could be the focus of a generalisation attempt. Among these are the symmetry of ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$ and corresponding conservation of angular momentum, gauge invariance (albeit restricted to plane gravitational waves) and satisfaction of the weak and dominant energy conditions. In this chapter we have explored only one such avenue: the total conservation of energy-momentum between matter and gravity in metrical GR. As we have emphasised already, the *metric* is not necessarily the fundamental dynamical variable of gravity, i.e. it may be that our failure compares to expanding a function $f(x)$ of no particular parity in x^2. We note that the use of tetrads in the nonlinearisation procedure is also suggested in [89]. There are two main points to take away from our analysis

1. In the harmonic gauge, the linearised pseudotensor of Einstein is equivalent to ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$ up to an identically conserved gauge current.
2. The original conservation law obeyed by ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$ does not admit a symmetric, third-order correction to ${}_{\mathrm{B}}\tau_{\mathfrak{ab}}$, quadratic in the first derivatives of the metric perturbation under the suggested perturbation schemes.

Given Item 2, the failure at fourth order is irrelevant, since the required conservation law no longer holds. It is only by relaxing the conditions on the form of the conservation law to include an affine connection that we are able to make progress, and in doing so the missing component converges very rapidly order-by-order on the Christoffel symbols. This ultimately brings us to Item 1 and the pseudotensor of Einstein.

It is now apparent, further to the work of [87], that the Klein–Gordon correspondence is a strong-field phenomenon in certain symmetric spacetimes, and that it applies to at least three formalisms for localising gravitational stress and energy. The strong-field extension is particularly interesting: the energetics of the gravitational field naturally identify a scalar field φ as the *gravitational potential*, which retains its simple Newtonian form above the surface of the densest neutron star (provided the star is not spinning). It has been shown (see [104]) that several energy-momentum complexes agree in a wide class of spacetimes under quasi-Cartesian coordinates, perhaps suggesting that the correspondence has a broader demographic than we have considered. Furthermore, the restriction to static spacetimes may prove unnecessary: isotropic coordinates, known as *Weyl's canonical coordinates*, are very useful in describing stationary axisymmetric spacetimes [105]. Conceivably, a generalisation of φ to stationary spacetimes might be reminiscent of gravitoelectromagnetism. A necessary part of the picture however, appears to be the flat background provided by the gauge theory.

Separately we have remarked on the relationship reflected in Fig. 1.2, that the conserved mass $\mathcal{M}_{\mathrm{T}}$ and the proper mass $\mathscr{M}_{\mathrm{T}}$ of relativistic stars appear to somehow 'mirror' each other across the gravitational mass M_{T}. Specifically we have observed:

1. The binding energies $\mathcal{M}_{\mathrm{B}}$ and $\mathscr{M}_{\mathrm{B}}$ correspond in the Newtonian limit.
2. The factor of $\rho + 3P$ in the Komar density is suggestive of a *local* virial theorem satisfied by $\mathcal{M}_{\mathrm{T}}$ rather than $\mathscr{M}_{\mathrm{T}}$ – the latter satisfies a *global* virial theorem.
3. On the M_4 background of GTG, certain gravitational stress-energy pseudotensors in certain spacetimes under isotropic coordinates imitate the SET of a scalar field φ, which appears to be generated by a gravitational mass density ϱ. This density replicates the same local virial theorem as the Komar density.
4. The energy budget of the same pseudotensors on M_4 takes the form $\mathcal{M}_{\mathrm{T}} + \mathcal{M}_{\mathrm{B}} = M_{\mathrm{T}}$.

Apart from the opposing sense in which the binding energy is 'lost' in either picture, it is worth noting that $\mathscr{M}_{\mathrm{B}}$ and $\mathcal{M}_{\mathrm{B}}$ can also differ significantly in magnitude, though it is not apparent from Fig. 1.2. For example, the binding energy of the Earth, which well approximates a Schwarzschild star, is revised either way by 1.2×10^8 g – nearly the mass of a blue whale. Our main contribution is Item 3 and the Klein–Gordon correspondence, though it is perhaps the least complete: the M_4 background is not observable, because all observers are bound by the gauge fields (equivalently, GTG is not an æther theory). Furthermore, the Klein–Gordon correspondence presently relies on a privileged isotropic coordinate system and we make no attempt in this chapter to construct a covariant generalisation. With this in mind, and pending such an investigation, it remains to be seen whether the quantities φ and ϱ may find some physical significance beyond the mathematical structure of GTG that suggests them.

In the next chapter we will transition over fully to the gauge theory approach, returning only briefly to metric-based theories in Chapter 3. In particular we will admit dynamical torsion by appending quadratic curvature invariants to the GTG action. We do not however persist with the use of geometric algebra. The GTG formulation can be easily adapted for the purposes of Chapters 2, 4 and 5, but from this point on such translations are confined to Appendices A.7 to A.9 while we proceed principally by means of tensors[19].

[19] Note also that we have seen the last of Penrose's slot notation.

Chapter 2

Poincaré gauge theory and emergent dark radiation

Abridged from W. E. V. Barker, A. N. Lasenby, M. P. Hobson and W. J. Handley, Physical Review D **102**, 024048 (2020), arXiv:2003.02690 [gr-qc].
Published content also appears in Appendices A.7, A.8 and B.1 to B.5.

2.1 Introduction

Once constrained by the strong cosmological principle (SCP), the geometry of the Universe is free to vary in two ways according to the Friedmann–Lemaître–Robertson–Walker (FLRW) metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{R^2\mathrm{d}r^2}{1-kr^2} - R^2r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2). \tag{2.1}$$

On the one hand *space*, defined by Cauchy surfaces containing material fluids at rest and spanned by dimensionless r, ϑ and φ, has curvature constant k equal to 1, 0 or -1. On the other *time*, here the dimensionful cosmic time t, distinguishes those same surfaces and parametrises the evolution of the dimensionful scale factor R along with derivative quantities such as the Hubble number H and deceleration parameter q

$$H \equiv \partial_t R/R, \quad q \equiv -R\partial_t^2 R/(\partial_t R)^2. \tag{2.2}$$

Einstein's GR predicts the geodesic trajectory of light, according to which recent measurements have been used to establish that at the present epoch the Universe is expanding, accelerating and either spatially flat or very large

$$H_0 > 0, \quad q_0 < 0, \quad |k|/(H_0^2R_0^2) \ll 1. \tag{2.3}$$

The cosmic concordance, or LCDM model [23], aims to reconcile these observations with the rest of GR, whose contemporary Friedmann equations can be written as

$$\mathsf{h}^2 = \omega_\mathrm{r} + \omega_\mathrm{m} + \omega_\Lambda + \omega_k, \tag{2.4a}$$

$$q_0\mathsf{h}^2 = \omega_\mathrm{r} + \frac{1}{2}\omega_\mathrm{m} - \omega_\Lambda. \tag{2.4b}$$

In these equations the Hubble number (or today's Hubble *constant*) is normalised to $\mathsf{h} \equiv H_0/\mathsf{H}$, where $\mathsf{H} \equiv 100\mathrm{km\,s^{-1}\,Mpc^{-1}}$, while a material (non-gravitational) density ρ_i gives rise to a contemporary dimensionless density according to

$$\Omega_{i,0} \equiv \kappa\rho_{i,0}/3H_0^2, \quad \omega_i \equiv \Omega_{i,0}\mathsf{h}^2. \tag{2.5}$$

In particular, radiation is only partly accounted for by the photons of the CMB $\omega_\mathrm{r} \equiv \big(1+\frac{7}{8}\big(\frac{4}{11}\big)^{4/3}N_\mathrm{eff}\big)\omega_\gamma$, with neutrinos making up the remaining relativistic D.o.F $N_\mathrm{eff} = N_{\nu,\mathrm{eff}}$. Matter, or pressureless dust, can be partitioned into its baryonic and CDM fractions $\omega_\mathrm{m} \equiv \omega_\mathrm{b} + \omega_\mathrm{c}$. Dark energy is assumed to emerge from a *gravitational* comsological constant Λ, which nonetheless fits into (2.5) with $\rho_\Lambda \equiv \Lambda/\kappa$; similarly the curvature density $\omega_k \equiv -k/R_0^2\mathsf{H}^2$ may notionally be fitted into (2.5) as $\rho_k \equiv -3kR_0^2/(\kappa R^2)$. The *deceleration equation* (2.4b) may be obtained from (2.4a) so long as the dependence of the various material energy-densities on R – their equations of state (E.o.S) $w_i \equiv P_i/\rho_i$ – are known. In particular, these are[1]

$$w_\mathrm{r} \equiv 1/3, \quad w_\mathrm{m} \equiv 0, \quad w_k \equiv -1/3, \quad w_\Lambda \equiv -1. \tag{2.6}$$

It is worth noting that the *energy balance* equation (2.4a) may be understood heuristically as a dimensionless statement of zero net energy density, in the sense that the Einstein tensor provides a formal and covariant notion of gravitational energy in GR, although we found in Chapter 1 that this picture remains deeply dissatisfying. Accordingly, we may write

$$\omega_\mathrm{r} + \omega_\mathrm{m} + \omega_\Lambda + \omega_H + \omega_k = 0, \tag{2.7}$$

where the final two dimensionless densities are strictly gravitational in origin: the accepted quantity ω_k conveys the energy stored in curled-up Cauchy surfaces, while we define

$$\omega_H \equiv -\mathsf{h}^2, \tag{2.8}$$

i.e. the 'kinetic energy density' of such surfaces as they expand or contract. Overall, (2.7) encodes a central tenet of modern cosmology: that R-evolution is fundamentally dependent on k.

Since its inception, many authors [41] have expressed concern with the LCDM model. In particular the required substances known as dark matter and dark energy remain unaccounted for, while the comparability of their densities at the present epoch is deemed so unlikely that it has become known as the *cosmic coincidence problem* [106]. Similarly, the *flatness problem* is to be resolved by bolting on a non-gravitational inflationary mechanism at early times [107]. While such long-standing objections stem from naturalness and Occam's razor, in recent years claims of observational inconsistencies with LCDM have become more common. These possible inconsistencies appear at homogeneous scales in the form of the *Hubble tension* [108] and *curvature tension* [45, 44], and affect structure formation through the *small scale crisis* [43]. The first of these is probably the most severe. At the far end of the cosmic distance ladder, major observational endeavours such as WMAP [109] and most recently Planck [24] have caused a low value of H_0 or h to be inferred from the CMB. More local measurements using

[1] In the case of gravitational quantities, we may infer an effective w_i from $\rho_i = \rho_i(R)$ and the Gibbs relation: the (intensive) work $\mathrm{d}\rho_i$ done by $\mathrm{d}R$ without heating.

Cepheid-calibrated supernovae data (SH0ES) [110], the tip of the red giant branch (TRGB) [111, 112], combined electromagnetic and gravitational observation of neutron star mergers [113], or multiply lensed quasar systems (H0LiCOW) [114] indicate a somewhat higher value. Moreover, there is a perception that this situation is exacerbated by each generation of experiments [46]. By one current estimate [48], the H_0 discrepancy has placed LCDM in jeopardy to the tune of 4.4σ.

In this chapter, we will motivate a modified gravity theory, the effect of which on the background cosmology can be packaged into an augmentation of LCDM, involving the addition of a small extra component ω_{eff}. The E.o.S parameter w_{eff} of this extra component 'tracks' the dominant cosmic fluid in (2.6), such that

$$w_{\text{r,eff}} \equiv 1/3, \quad w_{\text{m,eff}} \equiv (1-1/\sqrt{3})/2, \quad w_{\Lambda,\text{eff}} \equiv -1/\sqrt{3}. \tag{2.9}$$

Since $w_{\text{r,eff}} = w_{\text{r}}$, while $w_{\text{m,eff}} > w_{\text{m}}$ and $w_{\Lambda,\text{eff}} > w_{\Lambda}$, the extra component manifests an injection of *dark radiation* in the early Universe which redshifts away nontrivially at later times. In this sense, it can be cast as an extra relativistic species $N_{\text{eff}} = N_{\nu,\text{eff}} + \Delta N_{\text{dr,eff}}$. Similar models have recently become very popular [115–118] as a means to alleviate the H_0 tension. Some of these are in conflict with the observational constraints from Big Bang nucleosynthesis (BBN) or even from the CMB itself (see e.g. [119–123, 117]). Of greater concern is the reliance of many of these models on ad hoc physics.

In our case, the extra component picture is effective, since it emerges from a motivated modified gravity theory. Such alternatives to GR are themselves very popular, and may variously seek to cast early and late-time inflation as emergent gravitational phenomena, or conveniently resolve other tensions and crises in LCDM. A deeper motivation to modified gravity is the incompatibility of GR with quantum mechanics, and this provides further constraints on the theory. In particular GR is not perturbatively renormalisable, and modifications which fix this tend to do so at the expense of unitarity [49].

Amongst the modified gravity theories, the gauge theories have a heritage dating back to before the golden age of GR [124]. Rather than the internal $\text{SU}(3)_\text{c} \times \text{SU}(2)_\text{L} \times \text{U}(1)_\text{Y}$ group of the SM, comprising the strong and electroweak Glashow–Weinberg–Salam (GWS) forces, these theories gauge the assumed external symmetry group of spacetime. The diffeomorphism invariance of GR already encodes the gauged translational symmetry group $\mathbb{R}^{1,3}$ [15]. The least controversial extension ought to be such translations in combination with proper, orthochronous Lorentz rotations $\mathbb{R}^{1,3} \rtimes \text{SO}^+(1,3)$, which constitute the Poincaré group $\text{P}(1,3)$. This results in the Poincaré gauge theory (PGT) of Kibble [63], Utiyama [64] and Sciama [65]. Typical formulations of PGT split the metric into the square of a translational gauge field and introduce a rotational gauge field into the affine connection. This process introduces a geometric quality on the spacetime known as *torsion*, which is distinct from *curvature*. The spacetime is then said to be of Riemann–Cartan type U_4. A special case of PGT known as *teleparallelism*, in some sense antipodal to diffeomorphism gauge theories such as GR or $f(\mathcal{R})$ gravity, is reached by replacing curvature with torsion altogether – in this case the flat but twisted spacetime is of Weitzenböck type T_4 [67].

An expanded choice of symmetry group is that of Weyl $\text{W}(1,3)$. In this case, spacetime is symmetric under all elements of the extended conformal group $\text{C}(1,3)$ excluding special conformal transformations. As an extension to PGT this adds Weyl rescalings to the list of symmetries which need to be gauged, and results in Weyl gauge theory (WGT) on Weyl–Cartan spacetime Y_4 [125]. It is not entirely clear how the rotational gauge field should respond to Weyl rescalings, and WGT was recently *extended* (eWGT) [92] by promoting this freedom to an internal gauge symmetry (the so-called *torsion-scale gauge*). The relationship between PGT, WGT and eWGT is explained in detail in [92]. In a world with discrete mass spectra, it is accepted that the scale gauge symmetry, if present, must be broken. In WGT this is usually done explicitly (e.g. by fixing to the Einstein – sometimes called ‘unitary’ [126] – gauge), but it is possible to re-cast the equations of both WGT and eWGT in terms of scale-invariant variables which eliminate the scale gauge freedom and the need for explicit symmetry breaking. It is not clear that either method is preferable, or even that they differ in a physical sense.

A similar question, which develops out of our discussion in Section 1.3, surrounds the rôle of geometry in these gauge theories: it is possible to eliminate any combination of curvature, torsion and scale as geometric qualities of the spacetime in favour of field strengths on a spacetime without these qualities, finally arriving at gauge theory on Minkowski spacetime M_4. This raises serious questions only when topology is considered important[2]. For our purposes, we find the Minkowski interpretation to be the simplest basis for comparison between gauge theories. Consequently it is very important to note from this point onwards that we will use the term *Minkowski spacetime* quite loosely. In the first instance, we refer to the *kinematic state* adopted by various theories of gravity, in which all *geometrically interpretable* field strength tensors vanish. It should be borne in mind however that any of these theories may be formally cast in M_4 anyway, regardless of state, and incurring no contradiction.

As with diffeomorphism gauge theory, gauge theories in general enjoy a large freedom in their Lagrangian structure. Each gauged spacetime symmetry introduces a new field strength, but may impose restrictions on the field strength invariants appearing in the Lagrangian. Stable PGTs may be powered by a gravitational sector constructed from invariants of two gauge field strengths, the curvature tensor $\mathcal{R}_{ijkl}$ and torsion tensor $\mathcal{T}_{ijk}$. Since the successful SM relies on Yang–Mills gauge theories of internal symmetry groups, it is tempting to consider *quadratic* invariants of these tensors. The only linear invariant within PGT is the Ricci scalar $\mathcal{R}$ which alone constitutes the minimal gauge gravity extension to GR which is ECT. We refer to PGTs and eWGTs including all possible quadratic and linear invariants as PGT$^{\mathrm{q}}$ and eWGT$^{\mathrm{q}}$. Within PGT$^{\mathrm{q}}$ it is possible to roughly halve the dimensionality of the parameter space by imposing parity invariance on the gravitational sector, resulting in PGT$^{\mathrm{q}+}$ and, analogously, eWGT$^{\mathrm{q}+}$. This approach is commonly used in the literature, and constrains the theory in a natural manner. It must however be noted that a subset of authors (see e.g. [127]) reject it on the grounds of poor physical motivation.

Applications of gauge theory to cosmology began in the early 1970s and now constitute a large and established literature, with many authors progressing well beyond formalism to obtain analytical and numerical results. The earliest attempts narrowly focus on ECT, with the opening move being made by Kopczyński [128] who showed that the algebraic spin-torsion interaction could remove the singularity at the Big Bang. The modern notion of cosmological torsion in general, which we discuss in Section 2.4.2, was established by Tsamparlis [129] before the end of the decade. Full PGT$^{\mathrm{q}+}$ was incorporated by Minkevich in 1980 [130], who identified a set of generalised cosmological Friedmann equations which result from a *single* parameter constraint on the PGT$^{\mathrm{q}+}$ action. Minkevich has studied these equations in the context of singularity removal [131, 132], inflation [133] and dark energy [134, 135], see also [136]. These equations have also been analysed in the context of metric-affine gauge theory (MAGT) which lives on the linearly connected space L_4, a generalisation of U_4 which lacks the metricity condition [137]. The first thorough (and widely cited) exposition on the cosmology of PGT$^{\mathrm{q}+}$ was undertaken four years later by Goenner and Müller-Hoissen [138], although their examination of the parameter space was by no means exhaustive. For a comprehensive review of the literature prior to 2004, see [139]. In 2005

[2]For example a wormhole is difficult to cast in the Minkowski interpretation, as is the entire apparatus of Penrose diagrams.

an isolated study of pure Riemann-squared theory (RST) was undertaken [140]. Within $\text{PGT}^{\text{q}+}$, RST is a minimal quadratic alternative to ECT known to accommodate at least a Schwarzschild–de Sitter vacuum solution, and although the cosmological model suffers from scale-invariance – more specifically *normal scale-invariance* (NSI) – it admits emergent inflationary behaviour.

Superficially, these early classical endeavours may convey the impression that all emergent gravitational phenomena are available for free: questions raised by LCDM are simply absorbed into the fine-tuning of the ten PGT Lagrangian parameters. In 2008 quantum mechanical feasability entered in a seminal paper [141] by Shie, Nester and Yo (SNY), who observed that the 0^+ and 0^- torsional modes of PGT are naturally suited to cosmological investigation. Their $\text{PGT}^{\text{q}+}$ Lagrangian was constructed to target the 0^+ mode, and as such their quadratic Riemann–Cartan sector contains only $\mathcal{R}^2$. In the same year Li, Sun and Xi performed a numerical study of the system [142]. Chen, Ho, Nester, Wang and Yo later augmented their Lagrangian with the square pseudoscalar Riemann–Cartan term in order to include the 0^- mode [143]. Significant advances to the SNY Lagrangian were made in 2011 when Baekler, Hehl and Nester (BHN) included the parity-violating terms of PGT^{q} [127]. The cosmological implications of all parity-violating 'shadow world' terms and parity-preserving 'world' terms were distilled by means of cosmologically harmless parameter constraints into their representative BHN Lagrangian. This work was still being explored by the same authors in 2015, see [144–147]. Further work on the parity-preserving SNY Lagrangian was performed by Ao and Li in 2012 [146]. Most recently, Zhang and Xu (ZX) in [148, 149] have proposed a parameter constraint similar to that of Minkevich on $\text{PGT}^{\text{q}+}$ which suggests a pleasing inflationary formalism. We note that the apparent trend toward quadratic Lagrangia is not universal, as ECT remains popular to this day [150, 151] as a simple way to import torsion, albeit algebraically bound to spin. Moreover, other authors have considered cosmological models with torsion which do not quite fit into the PGT^{q} category, such as $f(\mathcal{R})$ and $\mathcal{R}^n$ PGTs, see for example [149].

The theoretical development of eWGT was first introduced to the community in 2016, and from the outset it has been clear that structure of eWGT has more in common with PGT than WGT (for a recent incorporation of scale-invariance to $\text{PGT}^{\text{q}+}$, see [126]). Indeed $\text{PGT}^{\text{q}+}$ and $\text{eWGT}^{\text{q}+}$ both sport ten Lagrangian parameters[3]. In this chapter, which represents the first application of eWGT to cosmology, we aim to show that $\text{PGT}^{\text{q}+}$ and $\text{eWGT}^{\text{q}+}$ are cosmologically equivalent.

The remainder of this chapter is structured as follows. In Section 2.2 we briefly explain the Minkowski interpretation of the two gauge theories under consideration, $\text{PGT}^{\text{q}+}$ and $\text{eWGT}^{\text{q}+}$, as in [92]. In Section 2.3 we review recent results concerning the quantum mechanics of $\text{PGT}^{\text{q}+}$, as contained within our major references [152, 153]. In Section 2.4 we adapt the minisuperspace formalism to $\text{PGT}^{\text{q}+}$ and $\text{eWGT}^{\text{q}+}$ cosmology and set out a cosmological correspondence between the actions of the two theories.

Our central results are confined to Section 2.5. The generalised Friedmann equations, which are common to $\text{eWGT}^{\text{q}+}$ and $\text{PGT}^{\text{q}+}$, are dissected in the context of quantum feasibility in Section 2.5.2, and the consequent k-screening in Section 2.5.3. The new cosmology behind (2.9) is then developed in Section 2.5.5 before we conclude in Section 2.6. There follows a list of the spin projection operators (SPOs) used for [152, 153] in Appendix B.1, a comparison to part of the literature mentioned above in Appendix B.4, and certain cumbersome functions in Appendix B.5.

[3]For this reason, we will not attend to WGT cosmology.

2.2 Gauge theories

2.2.1 Symmetries, transformation laws and field strengths

Gauge theories may be cast (almost) without loss of generality in a manifold $\check{\mathcal{M}}$ with Minkowskian geometry, as set out in Section 1.2.1. This Minkowski interpretation was pioneered by Kibble [63] (later reinterpreted for the STA by Lasenby and others [69]) and Blagojević [67], and used in the initial proposal for eWGT [92]. There is a potentially curvilinear coordinate system $\{x^\mu\}$ in this spacetime, with coordinates considered to be functions of the points of the manifold, and all fields written as functions of the coordinates. From the $\{x^\mu\}$ there is defined a basis of tangent vectors $\{\boldsymbol{e}_\mu\}$ and cotangent vectors $\{\boldsymbol{e}^\mu\}$ in the usual manner. The necessarily flat metric on $\check{\mathcal{M}}$, which is *not* a gravitational gauge field, is then $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = \gamma_{\mu\nu}$. Note that we drop the háček symbol denoting the Cartesian coordinates, but that replacing the tensor slots from Section 1.2.1 with our general coordinates we will have $\check{g}_{\mu\nu} = \gamma_{\mu\nu}$. The first gauge symmetry to consider is that of diffeomorphisms, though these are interpreted as passive general coordinate transformations (GCTs). Particularly, physical quantities should have zero *total* (as supposed to *form*) variations under GCTs. Taking new coordinates, $\{x'^\mu\}$, the covariance of a scalar matter field is expressed as

$$\varphi'(x') = \varphi(x), \tag{2.10}$$

with the expected transformation of other quantities

$$\boldsymbol{e}'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\boldsymbol{e}^\nu, \quad \boldsymbol{e}'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\boldsymbol{e}_\nu, \quad \partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu. \tag{2.11}$$

Independently of the coordinate basis, there exists an orthonormal Lorentz basis $\{\hat{\boldsymbol{e}}_i\}$ and dual basis $\{\hat{\boldsymbol{e}}^i\}$, such that $\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \eta_{ij}$. While Greek indices transform under the Jacobian matrices of GCTs, Roman indices transform under local Lorentz rotations $\Lambda^i{}_j$. Indices are converted by means of the translational gauge fields (analogous to the *tetrads* of the geometrical interpretation) $h_i{}^\mu$ and $b^i{}_\mu$, which themselves transform according to their indices[4]

$$h'_i{}^\mu = \Lambda^j{}_i \frac{\partial x'^\mu}{\partial x^\nu} h_j{}^\nu, \quad b'^i{}_\mu = \Lambda^i{}_j \frac{\partial x^\nu}{\partial x'^\mu} b^j{}_\nu, \tag{2.12}$$

and which satisfy $h_i{}^\mu b^i{}_\nu \equiv \delta^\mu_\nu$ and $h_i{}^\mu b^j{}_\mu \equiv \delta^j_i$. The matter field should of course be generalised to some higher-spin representation of the Lorentz group. A spacetime derivative on φ (scalar or otherwise), covariantised with respect to both gauge freedoms, can then be defined as

$$\mathcal{D}_i\varphi \equiv h_i{}^\mu\Big(\partial_\mu + \frac{1}{2}A^{kl}{}_\mu\Sigma_{kl}\Big)\varphi, \quad D_\mu\varphi \equiv b^i{}_\mu\mathcal{D}_i\varphi, \tag{2.13}$$

where $A^{ij}{}_\mu$ is the rotational gauge field and the Σ_{ij} are the $\mathrm{SO}^+(1,3)$ generators of the spin-specific representation of φ. Note that in this general representation the associated indices are suppressed. By convention, calligraphic script is used to highlight components of tensors defined purely with respect to the Lorentz frames, while normal script is used for mixed or purely coordinate frame definitions[5]. Thus,

[4]The gauge field determinant h should not be confused with the normalised Hubble constant, $\mathsf{h} \equiv H_0/\mathsf{H}$.

[5]This is especially useful in the present chapter, as we can always refer to $\mathcal{R}_{ijkl}$ instead of $R_{\rho\sigma\mu\nu}$, and thus avoid confusion with the dimensionful scale factor, R.

we note the required transformation properties of the spin connection under a pure Lorentz rotation

$$\mathcal{A}'^{ij}{}_{k} = \Lambda^{l}{}_{k}\left(\Lambda^{i}{}_{n}\Lambda^{j}{}_{p}\mathcal{A}^{np}{}_{l} - \Lambda^{jn}h_{l}{}^{\nu}\partial_{\nu}\Lambda^{i}{}_{n}\right), \quad \mathcal{A}^{ij}{}_{k} \equiv h_{k}{}^{\mu}A^{ij}{}_{\mu}, \tag{2.14}$$

The field strength tensors of PGT are then defined in the Yang–Mills sense

$$2\mathcal{D}_{[i}\mathcal{D}_{j]}\varphi = \Big(\frac{1}{2}\mathcal{R}^{kl}{}_{ij}\Sigma_{kl} - \mathcal{T}^{k}_{ij}\,\mathcal{D}_{k}\Big)\varphi, \tag{2.15}$$

where the Riemann–Cartan (rotational) field strength tensor is

$$\mathcal{R}^{ij}{}_{kl} \equiv 2h_{k}{}^{\mu}h_{l}{}^{\nu}\big(\partial_{[\mu|}A^{ij}{}_{|\nu]} + A^{i}{}_{m[\mu|}A^{mj}{}_{|\nu]}\big), \tag{2.16}$$

and the torsion (translational) field strength tensor is

$$\mathcal{T}^{i}_{kl} \equiv 2h_{k}{}^{\mu}h_{l}{}^{\nu}\big(\partial_{[\mu|}b^{i}{}_{|\nu]} + A^{i}{}_{m[\mu|}b^{m}{}_{|\nu]}\big). \tag{2.17}$$

Under local Weyl transformations, the various PGT quantities are expected to transform as

$$\varphi' = e^{w\rho}\varphi, \quad h'_{i}{}^{\mu} = e^{-\rho}h_{i}{}^{\mu}, \quad A'^{ij}{}_{\mu} = A^{ij}{}_{\mu}, \tag{2.18}$$

where w is the Weyl weight of the matter field. To arrive at WGT, the covariant derivative (2.13) must then be augmented with an extra Weyl gauge field. In eWGT, the spin connection obeys a more general transformation law

$$A'^{ij}{}_{\mu} = A^{ij}{}_{\mu} - 2\theta\eta^{k[i}b^{j]}{}_{\mu}h_{k}{}^{\nu}\partial_{\nu}\rho. \tag{2.19}$$

The dimensionless parameter[6] $\theta \in [0,1]$ is introduced to *extend* the *normal* transformation law of (2.18) to the *special* alternative, including admixtures between the two in its range[7]. The induced transformation of the PGT torsion contraction, $\mathcal{T}_{i} \equiv \mathcal{T}^{j}_{ij}$ and $T_{\mu} \equiv b^{i}{}_{\mu}\mathcal{T}_{i}$ combined with another θ-dependent transformation law for the Weyl gauge field

$$T'_{\mu} = T_{\mu} + 3(1-\theta)\partial_{\mu}\rho, \quad V'_{\mu} = V_{\mu} + \theta\partial_{\mu}\rho, \tag{2.20}$$

allows a suitable eWGT covariant derivative to then be constructed

$$\mathcal{D}^{\dagger}{}_{i}\varphi \equiv h_{i}{}^{\mu}\Big(\partial_{\mu} + \frac{1}{2}A^{\dagger jk}{}_{\mu}\Sigma_{jk} - wV_{\mu} - \frac{1}{3}wT_{\mu}\Big)\varphi, \quad D^{\dagger}{}_{\mu}\varphi \equiv b^{i}{}_{\mu}\mathcal{D}^{\dagger}{}_{i}\varphi, \tag{2.21}$$

In general, eWGT quantities are distinguished from their PGT counterparts by an obelisk superscript: the eWGT spin connection is $\mathcal{A}^{\dagger ij}{}_{k} \equiv \mathcal{A}^{ij}{}_{k} + 2\mathcal{V}^{[i}\delta^{j]}_{k}$, where $\mathcal{V}_{i} \equiv h_{i}{}^{\mu}V_{\mu}$. By generalising (2.13) to (2.21), the translational and rotational gauge field strengths are themselves redefined, and the extra gauge symmetry introduces its own field strength tensor

$$2\mathcal{D}^{\dagger}{}_{[i}\mathcal{D}^{\dagger}{}_{j]}\varphi = \Big(\frac{1}{2}\mathcal{R}^{\dagger kl}{}_{ij}\Sigma_{kl} - w\mathcal{H}^{\dagger}{}_{ij} - \mathcal{T}^{\dagger k}{}_{ij}\mathcal{D}_{k}\Big)\varphi. \tag{2.22}$$

[6]The parameter θ should not be confused with the polar angle ϑ.

[7]Note that although the special transformation is defined as $\theta = 1$, the apparatus of eWGT also functions outside the range $\theta \in [0,1]$.

In particular the eWGT Riemann–Cartan tensor differs from (2.16) according to

$$\mathcal{R}^{\dagger ij}{}_{kl} \equiv \mathcal{R}^{ij}{}_{kl} + 2\delta^{[j|}_{l}(\mathcal{D}_k + \mathcal{V}_k)\mathcal{V}^{|i]} - 2\delta^{[j|}_{k}(\mathcal{D}_l + \mathcal{V}_l)\mathcal{V}^{|i]} - 2\mathcal{V}^p\mathcal{V}_p\delta^{[i|}_{k}\delta^{|j]}_{l} + 2\mathcal{V}^{[i|}\mathcal{T}^{|j]}{}_{kl}\ , \tag{2.23}$$

while the eWGT torsion differs from (2.17) according to $\mathcal{T}^{\dagger i}{}_{jk} \equiv \mathcal{T}^{i}_{jk} + \frac{2}{3}\delta^{i}_{[j}\mathcal{T}_{k]}$, and has the property that its contractions vanish. We will not give the precise form of the field strength $\mathcal{H}^{\dagger}{}_{ij}$ associated with Weyl rescalings, since it is not used in the eWGT$^{\text{q}+}$ actions which follow on the grounds of potential instablility.

2.2.2 Restricted actions

The PGT$^{\text{q}+}$ Lagrangian density should be linear in gauge-invariant quantities with dimensions of energy density eV^4. Displacement gauge invariance naturally demands that these quantities be tensor densities of rank zero, while parity invariance further eliminates pseudoscalar densities. We are therefore interested in scalars, which we can always convert to densities by combination with the factor $b \equiv \det(b^{i}{}_{\mu}) \equiv h^{-1} \equiv 1/\det(h_{i}{}^{\mu})$. Within the gravitational sector, we are free to use invariants of the field strengths up to second-order. The only such first-order term is that of Einstein and Hilbert, which we write as[8] $L_{\mathcal{R}} \equiv -\frac{1}{2}\alpha_0\mathcal{R}$, where α_0 is a dimensionless parameter of the theory. Likewise, there are six such parameters in the quadratic Riemann–Cartan sector

$$L_{\mathcal{R}^2} \equiv \alpha_1\mathcal{R}^2 + \alpha_2\mathcal{R}_{ij}\mathcal{R}^{ij} + \alpha_3\mathcal{R}_{ij}\mathcal{R}^{ji} + \alpha_4\mathcal{R}_{ijkl}\mathcal{R}^{ijkl} + \alpha_5\mathcal{R}_{ijkl}\mathcal{R}^{ikjl} + \alpha_6\mathcal{R}_{ijkl}\mathcal{R}^{klij}, \tag{2.24}$$

and three more in the quadratic torsion sector

$$L_{\mathcal{T}^2} \equiv \beta_1\mathcal{T}_{ijk}\,\mathcal{T}^{ijk} + \beta_2\mathcal{T}_{ijk}\,\mathcal{T}^{jik} + \beta_3\mathcal{T}_{i}\,\mathcal{T}^{i}. \tag{2.25}$$

We also reserve the freedom at this stage to introduce an ad hoc cosmological constant $[\Lambda] = \text{eV}^2$. Anticipating various mechanisms which may give rise to an effective cosmological constant through the introduction of new dynamical fields, Λ will not be re-cast as a dimensionless theory parameter, and will enter into the Lagrangian as $L_\Lambda \equiv -\Lambda$. Finally, the various matter fields will couple to the gravitational gauge fields within their own Lagrangian densities: we will denote the resulting scalar simply as L_{M}. The general PGT$^{\text{q}+}$ action thus has ten dimensionless parameters, and by introducing Einstein's constant to compensate for dimensionality we may write it as

$$S_{\text{T}} \equiv \int \text{d}^4x h^{-1}\big[L_{\mathcal{R}^2} + \kappa^{-1}\left(L_{\mathcal{T}^2} + L_{\mathcal{R}} + L_\Lambda\right) + L_{\text{M}}\big]. \tag{2.26}$$

The situation for eWGT$^{\text{q}+}$ differs through the structure of the eWGT torsion tensor and the imposition of Weyl gauge invariance. The forms of $L_{\mathcal{R}^\dagger}$ and $L_{\mathcal{R}^{\dagger 2}}$ are identical to those of $L_{\mathcal{R}}$ and $L_{\mathcal{R}^2}$: one needs simply to replace the PGT Riemann–Cartan tensor with its eWGT counterpart. The quadratic torsion sector in eWGT$^{\text{q}+}$ contains only *two* D.o.F, because the eWGT torsion has identically vanishing contraction $L_{\mathcal{T}^{\dagger 2}} = \beta_1\mathcal{T}^{\dagger}{}_{ijk}\mathcal{T}^{\dagger ijk} + \beta_2\mathcal{T}^{\dagger}{}_{ijk}\mathcal{T}^{\dagger jik}$. The quadratic torsion and linear Riemann–Cartan sectors cannot be directly admitted to the Lagrangian because their Weyl weight is too low. This can be fixed by multiplication with a *compensator* field ϕ of dimension eV and weight $w = 1$, so that $\phi' = e^{\rho}\phi$. The generally dynamical nature of the compensator field demands the addition of an extra Lagrangian

[8] Note that in [92] the notation $\alpha_0 = a$ is used, which we will require for the dimensionless scale factor, $a \equiv R/R_0$.

contribution, which we write as a sum of kinetic and potential terms

$$L_\phi \equiv \frac{1}{2}\nu \mathcal{D}^\dagger{}_i\phi\mathcal{D}^{\dagger i}\phi - \lambda\phi^4. \tag{2.27}$$

The constraint on the Weyl weight of Lagrangian densities means that the second term in (2.27) already functions as a suitably general cosmological constant, therefore ν is the only new dimensionless theory parameter. A final possibility is a term quadratic in the Weyl gauge field strength $L_{\mathcal{H}^{\dagger 2}} \equiv \frac{1}{2}\xi\mathcal{H}^\dagger{}_{ij}\mathcal{H}^{\dagger ij}$, though in the present chapter we will take $\xi = 0$ as the field strength is incompatible with the SCP. Moreover, $\mathcal{H}^\dagger{}_{ij}$ has the unusual property of containing second derivatives of the $h_i{}^\mu$ gauge field: such a structure might be expected to introduce an Ostrogradsky instability to the E.o.M[9]. This may be compared to candidate terms in the PGT Lagrangian, quadratic in the first derivatives of the PGT torsion, which are traditionally excluded on similar grounds. The matter coupling will in general differ between eWGT and PGT, so we denote the matter Lagrangian by $L^\dagger{}_{\mathrm{M}}$ and write the total action as

$$S_{\mathrm{T}} \equiv \int \mathrm{d}^4x h^{-1}\left[L_{\mathcal{R}^{\dagger 2}} + \phi^2\left(L_{\mathcal{T}^{\dagger 2}} + L_{\mathcal{R}^\dagger}\right) + L^\dagger{}_{\mathrm{M}}\right]. \tag{2.28}$$

Note that while eWGT incorporates scale-invariance by guaranteeing homogeneous transformation of the covariant derivative $\mathcal{D}^\dagger{}_i$, some choices of PGT action are naturally scale-invariant despite the inhomogeneous transformation of $\mathcal{D}_i$. In the context of PGT$^{\mathrm{q}+}$, this holds for *normally* scale-invariant L_{M} in combination with $L_{\mathcal{R}} = L_{\mathcal{T}^2} = 0$, or the theory parameter constraint

$$\alpha_0 = \beta_1 = \beta_2 = \beta_3 = 0. \tag{2.29}$$

This imposes severe restrictions on both the gravitational sector, which is confined to the quadratic Riemann–Cartan sector, and the matter content, which is confined to radiation. We refer to such PGT$^{\mathrm{q}+}$s as *normally scale-invariant* (NSI).

In [92] it is noted that more general NSI versions of PGT$^{\mathrm{q}+}$ can be formed by allowing for the compensator ϕ field in PGT to make up for weights in both gravitational and matter sectors, as with eWGT. So long as no term proportional to $\mathcal{D}_i\phi\mathcal{D}^i\phi$ is is added to the matter sector, the constraints (2.29) on the gravitational sector can then be relaxed because the only remaining concern is the inhomogeneous transformation of $\mathcal{T}^i_{jk}$. This can be eliminated (up to a total derivative) by a restriction on the $\{\beta_M\}$

$$2\beta_1 + \beta_2 + 3\beta_3 = 0. \tag{2.30}$$

In what follows, as a matter of convenience, we will confine the ϕ field to eWGT.

We see therefore that the PGT$^{\mathrm{q}+}$ and eWGT$^{\mathrm{q}+}$ both contain ten freedoms at the level of the theory, and possibly an eleventh freedom in the form of the cosmological constant. There is some subtlety regarding the true freedom of the quadratic Riemann–Cartan sector in both cases, because of the Gauss–Bonnet identity, which states that the quantity

$$\mathcal{G} \equiv \mathcal{R}^2 - 4\mathcal{R}_{ij}\mathcal{R}^{ji} + \mathcal{R}_{ijkl}\mathcal{R}^{klij}, \quad \int \mathrm{d}^4x h^{-1}\mathcal{G} \equiv 0, \tag{2.31}$$

[9]We note however that there is some reason to believe that such problems may be self-resolving in practice [92].

is a total derivative in $d \leq 4$ dimensions, as is the analogous quantity in eWGT. This allows us to set one of α_1, α_3 or α_6 to zero without loss of generality. Since the invariance of physical results under a Gauss–Bonnet variation is a useful test, we will not make any such reduction for the purpose of simplifying calculations and instead maintain all six quadratic Riemann–Cartan parameters as far as possible.

Of greater relevance to the present chapter is the reparametrisation freedom under linear combinations: the $\{\alpha_I\}$, $\{\beta_M\}$ and ν are conveniently chosen to agree with the canonical form of tensor components. Unfortunately, this formulation does little to convey the effects of symmetry properties of the field strength tensors on the quadratic invariants. The symmetries of the Riemann–Cartan tensor are of fundamental importance when comparing these torsionful theories to more traditional metrical alternatives, and with this in mind we will work with the reparametrisation $\{\check{\alpha}_I\}$, $\{\check{\beta}_M\}$ provided in Eqs. (B.23d) to (B.23f). These parameters drop out of a new scheme for expressing quadratic invariants, which we set out in Appendix A.7. Note that as with β_3, the term parametrised by $\check{\beta}_3$ vanishes identically in $\mathrm{eWGT}^{\mathrm{q}+}$.

2.3 Ghosts, tachyons and loops

The perturbative QFT of $\mathrm{PGT}^{\mathrm{q}+}$ begins with the linearisation

$$h_i{}^{\mu} = \delta_i^{\mu} + f_i{}^{\mu}, \quad b^i{}_{\mu} = \delta^i_{\mu} - f^i{}_{\mu} + \mathcal{O}\left(f^2\right), \quad A^{ij}{}_{\mu} = \mathcal{O}\left(f\right). \tag{2.32}$$

The perturbative gravitational gauge fields with which we work are then[10]

$$f_{ij} = -\mathfrak{a}_{ij} - \mathfrak{s}_{ij}, \quad \mathfrak{a}_{ij} \equiv -f_{[ij]}, \quad \mathfrak{s}_{ij} \equiv -f_{(ij)}, \quad \bar{\mathfrak{s}}_{ij} \equiv \mathfrak{s}_{ij} - \frac{1}{2}\eta_{ij}\mathfrak{s}, \quad \mathfrak{s} \equiv -\bar{\mathfrak{s}} \equiv \mathfrak{s}^i{}_i, \tag{2.33}$$

i.e. two four-tensor fields of rank two and one of rank three. Upon canonical quantization, in composition with states of definite momentum or position, the four-tensor content of these fields will be distributed amongst states of definite spin-parity J^P. The J^P spectrum of any bosonic field is generally set by the rank n of the four-tensor, which is a tensor product of n four-vectors. Under a $\mathrm{SO}^+(1,3)$ rotation $\Lambda^i{}_j$ confined to a SO(3) rotation orthogonal to some timelike vector k^i, the timelike part of the four-vector transforms as a 0^+ state and the spacelike part as a 1^- state. Massive particle states are partly defined by a k^i in the form of their momentum, which can always be brought to a rest frame in which SO(3) is the Wigner little group, whose representations define these spins. Note that the spin picture breaks down in the massless case: the little group becomes SO(2), whose representations define no more than two *helicity* states. Carrying on, a rank-two four-tensor such as f_{ij} in (2.33) thus transforms as a state under the following equivalent representations of SO(3)

$$\left[\mathbf{D}(0^+) \oplus \mathbf{D}(1^-)\right] \otimes \left[\mathbf{D}(0^+) \oplus \mathbf{D}(1^-)\right] \simeq \mathbf{D}(0^+) \oplus \mathbf{D}(0^+) \oplus \mathbf{D}(1^-) \oplus \mathbf{D}(1^-) \oplus \mathbf{D}(1^+) \oplus \mathbf{D}(2^+), \tag{2.34}$$

indicating that the tensor is a direct sum of two 0^+, two 1^-, one 1^+ and one 2^+ states. An analogous calculation reveals that a general rank-three four-tensor is a direct sum of four 0^+, one 0^-, three 1^+, six 1^-, three 2^+, one 2^- and one 3^+ states. By adding the multiplicities of the states $2J+1$ for either field one recovers the 4^2 or 4^3 tensor D.o.F, illustrating the completeness of the J^P decomposition.

[10] Note that the quantities $\mathfrak{a}_{ij}$ and $\mathfrak{s}_{ij}$ in (2.33) are the *negative* of those used in [152, 153] since we prefer, following our conventions in Chapter 1, to use the $b^i{}_{\mu}$ and $g_{\mu\nu}$ perturbations over those of their inverses $h_i{}^{\mu}$ and $g^{\mu\nu}$.

In practice, the fields defined in (2.33) contain *a priori* symmetries which reduce their J^P content. In PGT, the antisymmetric part of f_{ij} introduces a 1^- and additional 1^+ sector to the theory, though both $\mathfrak{s}_{ij}$ and $\mathfrak{a}_{ij}$ excitations are always considered *gravitons*. The assumed antisymmetry of the spin connection $\mathcal{A}_{[ij]k} = \mathcal{A}_{ijk}$ eliminates three 0^+, one 1^+, four 1^- and two 2^+ sectors along with the 3^+ sector – excitations of the $\mathcal{A}^i{}_{jk}$ field are sometimes called *tordions* or *rotons*. In a general therefore, the gravitational particles of PGT remain maximally spin-2. It is worth noting that the distinction between symmetric and antisymmetric gravitons is rather artificial, as is the distinction between gravitons and tordions. This is because in many cases the various fields are related by gauge transformations or the excitations are coupled. The various J^P components of all fields may be extracted by means of SPOs, a full list of which is given in Appendix B.1.

We will use in Chapters 4 and 5 an alternative formalism for projecting out the J^P states, based on the ADM decomposition. The ADM projections use the time-foliation vector n^i rather than the timelike momentum k^i defined above. It will also become clear in Chapter 4 that one 0^+ and one 1^- mode in f_{ij}, and one 1^- and one 1^+ in $\mathcal{A}^{ij}{}_k$ are *discounted* from the dynamics, because of the ten Poincaré gauge generators.

The final theory parameters employed in [153] differ from those in [92] chiefly through mixing of the linear Riemann–Cartan and quadratic torsion sectors[11]. These are set out in Eqs. (B.23g) to (B.23i). In terms of these parameters, [152, 153] analyse the viability of the free-field theory from the perspective of the physical propagator. Also known as the saturated propagator, this quantity can be obtained when the SPO decomposition of the free-field action is expressible in terms of *invertible* matrices which quadratically combine the $\mathfrak{s}_{ij}$, $\mathfrak{a}_{ij}$ and $\mathcal{A}_{ijk}$ fields within each J^P sector. As might be expected, there exist certain *critical cases* for which some of these matrices become singular. Each such case is defined by certain equations which are linear in the parameters, and represents one or more emergent gauge symmetries in the linearised theory which must be eliminated before proceeding. Beyond such gauge symmetries, further critical cases alter the factorised form of the matrix determinants, which encode the bare mass spectrum of each J^P sector.

In [152], the 1918 such critical cases of $\mathrm{PGT}^{\mathrm{q}+}$ were exhaustively determined. A systematic survey of these theories identified the 450 for which unitarity can be achieved through additional inequality constraints on the parameters. This requires the elimination of ghost modes by fixing a positive propagator residue about the relevant pole, and tachyonic particles by fixing a positive square of the relevant bare mass. This survey followed earlier studies by Neville [154, 155], and later Sezgin and van Nieuwenhuizen who found a total of 12 such unitary cases [156, 157]. Any of these critical cases can be discarded if a power counting shows that the superficial degree of divergence in a diagram scales with the number of loops, as in Fig. 1. In [152], such an analysis was restricted to cases in which the propagator was diagonal not only in the J^P sectors, but also in the fields themselves. This yielded 10 cases which were power counting renormalisable (PCR).

Although the PCR condition is thought to be necessary (albeit insufficient) for actual renormalisability, it raises ambiguities when applied to $\mathrm{PGT}^{\mathrm{q}+}$. Firstly, there may be two or three gauge choices which eliminate the symmetries of a critical case, of which not all are PCR. Secondly, a mode with unsatisfactory high-energy behaviour may yet be non-propagating, and thus inconsequential. Such modes tend to arise precisely when the propagator is non-diagonal in the fields, in particular when the 1^+ and 1^- sectors

[11] Note that in [152] the Gauss–Bonnet identity is used to eliminate $\check{\alpha}_1$, which we resurrect through r_6, and the notation $l = \lambda$ is used, which we will require for the effective cosmological constant in eWGT, $\kappa^{-1}\Lambda = \lambda\phi_0^4$.

of $\mathcal{A}_{ijk}$ are mixed. Of the 450 unitary cases, a further 48 (not including the 10 cases from [152]) were found in [153] which can be considered PCR according to these extended criteria. In the present chapter, we exclude from all 58 theories only those for which the divergence of non-propagating modes is most egregious[12], going as k^2 rather than k^{-2}. This leaves us with 33 critical cases, which include all of the original 10 in [152]. These are listed in Table 2.1. Note that while the methods in [152, 153] can identify the *definite* J^P sectors of propagating massive modes, it can only identify the *possible* J^P sectors of propagating massless modes, and their *definite* D.o.F. In the remainder of this thesis, we will adhere to the numbering of critical cases used in [153], in which the select 33 cases we consider range from Case 1 to Case 41. We also use the convention of [153] in which cases previously discovered in [152] are listed with their original numbering in a superscript, such as Case ^{*1}9, Case ^{*3}10, Case ^{*4}11 and Case ^{*2}13, which are the only four cases with gauge-invariant PCR.

2.4 The cosmological ansatz

2.4.1 Lessons out of superspace

The E.o.M of a field theory are usually obtained using the Lagrangian, or less commonly the Hamiltonian – we will more substantially develop these in Chapter 3 and Chapter 4, respectively. In the theories of (potentially) high-spin fields such as those of gravity considered here, this process is typically lengthy and necessarily results in tensor equations. Once the gravitational field equations are to hand, it is most convenient either to solve the fields for a desirable source, or vice versa. In cases where the solutions are known to be highly symmetric, a suitable ansatz for both sources and fields may be substituted and these solved simultaneously: this is often done in cases where the SCP applies[13]. We take a short cut by substituting the source and field ansatz into the action directly, and taking variations with respect to the remaining free functions. It should be stressed that this method is *not* always justifiable, as variable reduction and variational differentiation are generally non-commuting operations. Nor is it entirely without precedent. In the quantum cosmology of GR, similar methods are frequently employed as part of the minisuperspace approximation [159]. Moreover, the approach has been shown to hold true in GR for all Bianchi A class cosmological models [160] and similar methods are even employed for PGT^{q+} in [143]. Special care must be taken, so that the field ansatz preserves some notion of the ADM lapse and shift freedoms, and that the source ansatz comes pre-packaged with the expected conservation laws [161, 162]. In this way, we can avoid intermediate tensor expressions, arriving at an unorthodox but useful statement of the general cosmological equations. These will be given in Eqs. (2.51a) to (2.51d).

2.4.2 Gravitational fields

The first task is to find the most general ansatz for each of the four gauge fields $h_i{}^\mu$, $b^i{}_\mu$, $A^{ij}{}_\mu$ and V^μ consistent with the strong cosmological constraints of spatial homogeneity and isotropy. These constraints do not apply directly to the gauge fields, but to the observable quantities derived from them. It is convenient to adopt spherical polar coordinates $\{x^\mu\} = \{t, r, \vartheta, \varphi\}$ where the only dimensionful coordinate is t. This fixes the diffeomorphism gauge via the basis vectors $\{\boldsymbol{e}_\mu\}$ and covectors $\{\boldsymbol{e}^\mu\}$.

[12]While this is probably a conservative move, it is foremost a matter of convenience. We also note that with the exception of Case ^{*1}9, Case ^{*3}10, Case ^{*4}11 and Case ^{*2}13 as labelled in [153], J^P sectors propagate which violate these rules. However, these 'bad' modes are understood to decouple at high energies without producing divergent loops [158].

[13]It is worth noting that an alternative 'intrinsic' method of solution has been developed for the GTG version of ECT from Chapter 1 [69].

Table 2.1 The select 33 of the unitary, PCR critical cases of PGT^{q+}, according to parameter constraints and particle content. The given numbers are as in [153], with the original numbers in [152] denoted by an asterisk where applicable. The criticality equalities include an implicit $r_6 = 0$, see Eqs. (B.23g) to (B.23i) for the coupling translations. The particle content of each J^P sector is as follows. Possible massless excitations of $\mathcal{A}_{ijk}$, $\mathfrak{s}_{ij}$ and $\mathfrak{a}_{ij}$ are respectively '○ (blue)', '○ (green)' and '○ (red)'. Definite massive excitations are '● (blue)', '● (green)' and '● (red)'. Possible massless excitations may have their field character transmuted by gauge transformations, e.g. '○–○ (blue–green)', or be of uncertain field character (coupled) in one or more such gauge, e.g. '○–○ (blue–blue/red)'. While the J^P character of propagating massless excitations remains ambiguous (because poles from multiple J^P sectors coincide at the origin of momentum-space), there are always two, if any, massless D.o.F.

(Symbols in the table cells: ● = filled blue dot; b = blue ring; g = green ring; r = red ring; br = half-blue, half-red ring; rings joined by – are linked. In the d.o.f column, ○ = grey ring.)

#	criticality equalities	ghost-tachyon exorcism inequalities	0^-	0^+	1^-	1^+	2^-	2^+	d.o.f
1	$l = r_1 = t_1 = t_3 = r_3 - 2r_4 = 0$	$0 < t_2, r_2 < 0, r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$	●		b	b–br		b	● ○ ○
2	$l = r_1 = t_1 = r_3 - 2r_4 = 0$	$0 < t_2, r_2 < 0, r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$	●	b–g	b–g–br	b–br		b	
8	$l = r_2 = r_4 = t_1 = t_2 = r_1 - r_3 = 0$	$r_1(r_1 + r_5)(2r_1 + r_5) < 0$		b–g	b–g–br	b	b		○ ○
*[1]9	$l = r_2 = r_4 = t_1 = t_2 = t_3 = r_1 - r_3 = 0$	$r_1(r_1 + r_5)(2r_1 + r_5) < 0$			b	b	b		
*[3]10	$l = r_1 = r_2 = t_1 = t_2 = t_3 = r_3 - 2r_4 = 0$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$				b		b	
*[4]11	$l = r_1 = t_1 = t_2 = t_3 = r_3 - 2r_4 = 0$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$	b		b	b		b	
12	$l = r_1 = r_2 = t_1 = t_3 = r_3 - 2r_4 = 0$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$	b		b	b–br		b	
*[2]13	$l = r_2 = t_1 = t_2 = t_3 = 2r_1 - 2r_3 + r_4 = 0$	$0 < r_1(r_1 - 2r_3 - r_5)(2r_3 + r_5)$		b	b	b	b		
14	$l = r_1 = r_2 = t_1 = t_2 = r_3 - 2r_4 = 0$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$		b–g	b–g–br	b		b	
15	$l = r_1 = r_2 = t_1 = r_3 - 2r_4 = 0$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$	b	b–g	b–g–br	b–br		b	
16	$l = r_1 = t_1 = t_2 = r_3 - 2r_4 = 0$	$r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$	b	b–g	b–g–br	b		b	
20	$l = r_1 = r_3 = r_4 = r_5 = 0$	$0 < t_2, r_2 < 0$	●	b–g	b–g–br	b–br	b	b–g	●
21	$l = r_1 = r_3 = r_4 = r_5 = t_1 + t_2 = 0$	$r_2 < 0, t_1 < 0$	●	b–g	b–g–br	b–br	b	b–g	
22	$l = r_1 = r_3 = r_4 = r_5 = t_1 + t_3 = 0$	$0 < t_2, r_2 < 0$	●	b–g	b–g–br	b–br	b	b–g	
23	$l = r_1 = r_3 = r_4 = r_5 = t_1 + t_2 = t_1 + t_3 = 0$	$r_2 < 0, t_1 < 0$	●	b–g	b–g–br	b–br	b	b–g	
24	$l = r_1 = r_3 = r_4 = t_1 = 0$	$0 < t_2, r_2 < 0$	●	b–g	b–g–br	b–br			
*[5]25	$l = r_1 = r_3 = r_4 = r_5 = t_1 = 0$	$0 < t_2, r_2 < 0$	●	b–g	b–g–r	b–r			
*[6]26	$l = r_1 = r_3 = r_4 = r_5 = t_1 = t_3 = 0$	$0 < t_2, r_2 < 0$	●			b–r			
27	$l = r_1 = t_1 = t_3 = r_3 - 2r_4 = r_3 + 2r_5 = 0$	$0 < t_2, r_2 < 0$	●			b–br		b	
28	$l = r_1 = r_3 = r_4 = t_1 = t_3 = 0$	$0 < t_2, r_2 < 0$	●		b	b–br			
29	$l = r_4 = t_1 = r_1 - r_3 = 2r_1 + r_5 = 0$	$0 < t_2, r_2 < 0$	●	b–g	b–g–br	b–r	b		
*[7]30	$l = r_4 = t_1 = t_3 = r_1 - r_3 = 2r_1 + r_5 = 0$	$0 < t_2, r_2 < 0$	●		b	b–r	b		
*[8]31	$l = r_1 = t_1 = t_3 = 2r_3 - r_4 = 2r_3 + r_5 = 0$	$0 < t_2, r_2 < 0$	●	b		b–r			
32	$l = r_1 = r_3 = r_4 = r_5 = t_3 = 0$	$0 < t_2, r_2 < 0$	●		b–g–r	br–br	b	b–g	
33	$l = r_1 = r_3 = r_4 = r_5 = t_3 = t_1 + t_2 = 0$	$r_2 < 0, t_1 < 0$	●		b–g–r	br–br	b	b–g	
34	$l = r_1 = t_1 = t_3 = 2r_3 - r_4 = 0$	$0 < t_2, r_2 < 0$	●	b	b	b–br			
*[9]35	$l = r_1 = t_1 = t_3 = r_3 - 2r_4 = 2r_3 + r_5 = 0$	$0 < t_2, r_2 < 0$	●		b	b–r		b	
*[10]36	$l = t_1 = t_3 = 2r_3 + r_5 = 2r_1 - 2r_3 + r_4 = 0$	$0 < t_2, r_2 < 0$	●	b	b	b–r	b		
37	$l = r_1 = t_1 = r_3 - 2r_4 = 2r_3 + r_5 = 0$	$0 < t_2, r_2 < 0$	●	b–g	b–g–br	b–r		b	
38	$l = r_1 = t_3 = 2r_3 - r_4 = 2r_3 + r_5 = 0$	$0 < t_2, r_2 < 0$	●	b	b–g–r	b–br	b	b–g	
39	$l = r_1 = t_3 = 2r_3 - r_4 = 2r_3 + r_5 = t_1 + t_2 = 0$	$r_2 < 0, t_1 < 0$	●	b	b–g–r	b–br	b	b–g	
40	$l = r_1 = t_1 = t_3 = r_4 + r_5 = 0$	$0 < t_2, r_2 < 0$	●	b		b–br		b	
41	$l = r_1 = t_1 = r_3 - 2r_4 = r_3 + 2r_5 = 0$	$0 < t_2, r_2 < 0$	●	b–g	b–g–r	b–br		b	

By orthogonality, the normalised counterparts of these eight quantities provide a natural choice of Lorentz rotation gauge $\{\hat{\boldsymbol{e}}_i\}$ and $\{\hat{\boldsymbol{e}}^i\}$, should we choose to fix it. An interval which suitably generalises (2.1) is then

$$\mathrm{d}s^2 = S^2\left[\mathrm{d}t^2 - \frac{R^2\mathrm{d}r^2}{1-kr^2} - R^2r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\right], \tag{2.35}$$

where $S = S(t)$ is a dimensionless conformal factor which establishes the length scale of the theory, $R = R(t)$ is the dimensionful relative scale factor while the constant $k \in \{0, \pm 1\}$ dictates the curvature of Cauchy surfaces. Note that setting $S = 1$ corresponds to the Friedmann gauge, in which R becomes the usual scale factor of the Universe. The interval (2.35) determines the components $b^i{}_\mu$ only up to the rotation gauge, which we leave arbitrary. The gauge fields are then fixed to

$$b^i{}_t = S(\hat{\boldsymbol{e}}_t)^i, \quad b^i{}_r = \frac{SR}{\sqrt{1-kr^2}}(\hat{\boldsymbol{e}}_r)^i, \quad b^i{}_\vartheta = rSR(\hat{\boldsymbol{e}}_\vartheta)^i, \quad b^i{}_\varphi = rSR(\hat{\boldsymbol{e}}_\varphi)^i, \tag{2.36}$$

up to a choice of sign. In practice, we will work exclusively with the inverse fields $h_i{}^\mu$. As $h_i{}^\mu$ has thus been determined by a cosmological $g_{\mu\nu}$, so $A^{ij}{}_\mu$ must be determined by a cosmological $\mathcal{T}^i{}_{jk}$. The unique form adopted by the torsion tensor under the restrictions of homogeneity and isotropy may be written down immediately (though we show it rigorously in Appendix B.2)

$$\mathcal{T}^i{}_{jk} = (\hat{\boldsymbol{e}}_t)^l\Big(\frac{2}{3}U\delta^i{}_{[k|}\eta_{l|j]} - Q\epsilon^i{}_{ljk}\Big), \tag{2.37}$$

where the fields $U = U(t)$ and $Q = Q(t)$ have dimensions of eV and are observable quantities which may be extracted through the quadratic invariants

$$\mathcal{T}^i\mathcal{T}_i = U^2, \quad \mathcal{T}_{ijk}\,\mathcal{T}^{jik} = \frac{1}{3}U^2 + 6Q^2, \quad \mathcal{T}_{ijk}\,\mathcal{T}^{ijk} = \frac{2}{3}U^2 - 6Q^2. \tag{2.38}$$

We will occasionally refer to the fields U and Q as the *torsion contraction* and *torsion protraction*, respectively – the reference to the protraction will be explained in Appendix A.7. Furthermore, we show in Appendix B.1 that the SCP has done nothing more than pick the 0^- and 0^+ sectors out of the general torsion tensor, so U and Q encode the freedoms in the scalar and pseudoscalar tordion singlets. From (2.38) we see right away that there is some degeneracy among the dimensionless theory parameters $\{\beta_M\}$ under cosmological conditions. This behaviour is to be expected, and is even more pronounced in the quadratic Riemann–Cartan sector: we will make extensive use of it in Section 2.5.

For the purposes of the ansatz, we take the torsion tensor to have the form

$$\mathcal{T}^i{}_{jk} = \frac{2}{SR}(\hat{\boldsymbol{e}}_t)^l\left[\left(X + \frac{\partial_t(SR)}{S}\right)\delta^i{}_{[k|}\eta_{l|j]} - \frac{Y}{2}\epsilon^i{}_{ljk}\right]. \tag{2.39}$$

The dimensionless fields $X = X(t)$ and $Y = Y(t)$ now inherit the two D.o.F in U and Q. The form of the first term in (2.39) is designed to absorb those Ricci rotation coefficients which contain $\partial_t S$ and $\partial_t R$, and the rotational gauge fields which generate this torsion are

$$A^{ij}{}_r = \frac{1}{\sqrt{1-kr^2}}(\hat{\boldsymbol{e}}_t)^k(\hat{\boldsymbol{e}}_r)^l\left(2X\delta^i{}_{[l}\delta^j{}_{k]} + Y\epsilon^{ij}{}_{kl}\right), \tag{2.40a}$$

$$A^{ij}{}_\vartheta = 2(\hat{\boldsymbol{e}}_\vartheta)^k\left[\frac{1}{r}\left(1-\sqrt{1-kr^2}\right)(\hat{\boldsymbol{e}}_r)^l + X(\hat{\boldsymbol{e}}_t)^l\right]\delta^i{}_{[k}\delta^j{}_{l]} + Y(\hat{\boldsymbol{e}}_t)^k(\hat{\boldsymbol{e}}_\vartheta)^l\epsilon^{ij}{}_{kl}, \tag{2.40b}$$

$$A^{ij}{}_{\varphi} = 2(\hat{\boldsymbol{e}}_{\varphi})^k \left[\frac{1}{r}\left(1-\sqrt{1-kr^2}\right)(\hat{\boldsymbol{e}}_r)^l + X(\hat{\boldsymbol{e}}_t)^l\right]\delta^i{}_{[k}\delta^j{}_{l]} + Y(\hat{\boldsymbol{e}}_t)^k(\hat{\boldsymbol{e}}_{\varphi})^l\epsilon^{ij}{}_{kl}, \tag{2.40c}$$

The E.o.M are therefore to be obtained through variation with respect to R, S, X and Y, yet the cosmological equations are ideally expressed in terms of observable quantities. Clearly S is not observable, because after variation we would like to adopt the Friedmann gauge by globally setting $S = 1$. Having done this, we also note that R is not generally a quantity with good physical motivation, since when $k = 0$ it may be chosen arbitrarily. With this in mind, we prefer to substitute for R, X and Y in terms of the Hubble number and deceleration parameter defined in (2.2), and physical torsion fields once the Friedmann gauge has been adopted

$$U = \frac{3}{R}\left(X + \partial_t R\right), \quad Q = \frac{Y}{R}. \tag{2.41}$$

Having established the gravitational field ansatz in PGT, the extension to eWGT is quite straightforward. The compensator, ϕ, naturally satisfies the SCP as a scalar field, $\phi = \phi(t)$. The minimal choice for the Weyl gauge field is then to define a dimensionless $V = V(t)$ such that $\mathcal{V}^i = V(\hat{\boldsymbol{e}}_t)^i/SR$.

2.4.3 Gravitational sources

We consider four distinct sources in our models, though three of these may correspond to a variety of physical matter fields. Firstly the curvature constant k is embedded in the gravitational rather than matter sector of the action, yet as we discussed in Section 2.1, it has become acceptable to view it as a source term in the cosmological equations. Dark energy, or vacuum energy is included via the cosmological constant Λ in $\text{PGT}^{\text{q}+}$ and parameter λ in $\text{eWGT}^{\text{q}+}$, and is already a valid cosmological source having both homogeneity and isotropy. Directly observable baryonic matter and CDM are modelled by *dust*, while photons and neutrinos are modelled by *radiation*. In making these approximations we forfeit any effects arising from the spin content of the real sources, but avoid the complexities of constructing Weyssenhoff fluids[14].

In establishing the form of L_{M} and $L^{\dagger}{}_{\text{M}}$, we adopt the techniques set out in [162, 166], taking the Lagrangian densities to be the negative on-shell energy densities of the fluids,

$$L_{\text{M}} = -\rho_{\text{m}} - \rho_{\text{r}} = -\frac{\varrho_{\text{m}}}{\sqrt{\kappa}S^3R^3} - \frac{\varrho_{\text{r}}}{S^4R^4}, \quad L^{\dagger}{}_{\text{M}} = -\sqrt{\kappa}\phi\rho_{\text{m}} - \rho_{\text{r}} = -\phi\frac{\varrho_{\text{m}}}{S^3R^3} - \frac{\varrho_{\text{r}}}{S^4R^4}, \tag{2.42}$$

where $\rho_{\text{m}} = \rho_{\text{m}}(t)$ and $\rho_{\text{r}} = \rho_{\text{r}}(t)$ have dimension eV^4 and ϱ_{r} and ϱ_{m} are dimensionless constants. As with the gravitational variables, we will prefer to express the matter content in the cosmological equations in terms of observable quantities. The constants Λ and k along with the densities ρ_{m} and ρ_{r} are already acceptable from this perspective, but we will make use of the popular *dimensionless densities* as they are defined in the Friedmann gauge,

$$\Omega_k \equiv -\frac{k}{R^2H^2}, \quad \Omega_\Lambda \equiv \frac{\Lambda}{3H^2}, \quad \Omega_{\text{m}} \equiv \frac{\kappa\rho_{\text{m}}}{3H^2}, \quad \Omega_{\text{r}} \equiv \frac{\kappa\rho_{\text{r}}}{3H^2}. \tag{2.43}$$

[14] Note that if the Dirac Lagrangian is rendered scale-invariant by means of the compensator ϕ, the resulting matter SET resembles that of a perfect fluid [163–165].

These quantities are well suited to the analysis that follows in Section 2.5, but differ from the contemporary densities in Section 2.1, which are typically used in the field of cosmological inference, through the normalisation of H according to (2.5).

2.5 General cosmologies

2.5.1 A demonstration: Einstein–Cartan theory

The E.o.M are to be obtained by considering PGT$^{\text{q}+}$ and eWGT$^{\text{q}+}$ actions of the form

$$\tilde{S}_{\text{T}} = \int \mathrm{d}t\tilde{\mathcal{L}}_T(X(t), Y(t), S(t), R(t)), \quad \tilde{S}_{\text{T}} = \int \mathrm{d}t\tilde{\mathcal{L}}_T(X(t), Y(t), S(t), R(t), \phi(t), V(t)). \tag{2.44}$$

To check the efficacy of our approach, we will obtain the Friedmann equations from the minimal gravitational gauge theory in which the $\{\check{\alpha}_I\}$ and $\{\check{\beta}_M\}$ are all set to zero except for $\check{\alpha}_0$: this is ECT. The PGT action S_{T} in (2.26) is the integral of the *reduced* action, $\tilde{S}_{\text{T}}$, over the Cauchy-surface

$$\tilde{S}_{\text{T}} = -\int \mathrm{d}t\big[3\check{\alpha}_0\kappa^{-1}S^2R\left(R\partial_tX + Y^2/4 - X^2 - k\right) + \kappa^{-1}\Lambda S^4R^3 + \varrho_{\text{m}}S/\sqrt{\kappa} + \varrho_{\text{r}}/R\big]. \tag{2.45}$$

There are four dynamical fields: two for curvature, R and S, and two for torsion, X and Y. It is with respect to these quantities, rather than their physical counterparts, that we must take variations. Once we set $S = 1$, the E.o.M for X and Y are

$$\left(\delta\tilde{\mathcal{L}}_{\text{T}}/\delta X\right)_{\text{F}} \propto R\left(\partial_t R + X\right), \quad \left(\delta\tilde{\mathcal{L}}_{\text{T}}/\delta Y\right)_{\text{F}} \propto RY, \tag{2.46}$$

which immediately confirms that cosmic torsion is prohibited in an ECT Universe filled with the simplistic source fluids considered here, or $U = Q = 0$. The curvature equations for R and S are

$$\left(\delta\tilde{\mathcal{L}}_{\text{T}}/\delta R\right)_{\text{F}} \propto 3\check{\alpha}_0R^2\left(2R\partial_tX - X^2 + Y^2/4 - k\right) + 3R^4\Lambda - \kappa\varrho_{\text{r}}, \tag{2.47a}$$

$$\left(\delta\tilde{\mathcal{L}}_{\text{T}}/\delta S\right)_{\text{F}} \propto 6\check{\alpha}_0R\left(R\partial_tX - X^2 + Y^2/4 - k\right) + 4R^3\Lambda - \kappa^{\frac{1}{2}}\varrho_{\text{m}}. \tag{2.47b}$$

The four Eqs. (2.46), (2.47a) and (2.47b) may then be re-arranged in terms of the preferred variables to give the cosmic E.o.M $\check{\alpha}_0 = \Omega_{\text{m}} + \Omega_{\text{r}} + \Omega_\Lambda + \Omega_k$ and $\check{\alpha}_0 q = \frac{1}{2}\Omega_{\text{m}} + \Omega_{\text{r}} - \Omega_\Lambda$. The Friedmann equations are recovered when we choose $\check{\alpha}_0 = 1$, thus making the connection to ECT.

The reduced action in eWGT naturally takes a very similar form to (2.45)

$$\tilde{S}_{\text{T}} = -\int \mathrm{d}t\big[3\check{\alpha}_0\phi^2S^2R\big(R\partial_t(X+V) + Y^2/4 - (X+V)^2 - k\big) + \lambda\phi^4S^4R^3 + \varrho_{\text{m}}\phi S + \varrho_{\text{r}}/R\big], \tag{2.48}$$

the important difference being the appearance of the ϕ field, which always appears in the combination ϕS, and the V field, which appears in the combination $X + V$. These are perfectly general features of cosmological eWGT$^{\text{q}+}$: the extra gauge fields are degenerate with two of the original four in PGT$^{\text{q}+}$

$$\phi \leftrightharpoons S, \quad V \leftrightharpoons X. \tag{2.49}$$

The degeneracy (2.49) clearly indicates that we will have no more independent E.o.M in eWGT^{q+} than in PGT^{q+}, but the fixing of the Friedmann gauge in the former case remains to be defined. In particular, V can be absorbed directly into X since both fields are dimensionless. Finally, if the fixing of $S=1$ is carried over to eWGT^{q+}, we find the appropriate Einstein gauge $\phi=\phi_0=1/\sqrt{\kappa}$ completes the correspondence. Note that in this case, the freedom in Λ is truly inherited by the dimensionless λ rather than ϕ.

2.5.2 The cosmic theory parameters

We would now like to consider the general actions of PGT^{q+} and eWGT^{q+}, (2.26) and (2.28). The parameter degeneracy among the torsion variables identified in (2.38) extends throughout the gravitational sector, allowing us to express the E.o.M minimally in terms of parameter combinations which uniquely affect the cosmology. It is expedient to use vector notation to discuss theories, for example any PGT^{q+} may be written in terms of its theory parameters as $\boldsymbol{x}=\sum_{I=0}^{6}\check{\alpha}_I\check{\boldsymbol{\alpha}}_I+\sum_{M=1}^{3}\check{\beta}_M\check{\boldsymbol{\beta}}_M$, such that the vectors on the RHS form an orthonormal set, and any theory parameter may be extracted by projecting with the relevant vector, e.g. $\check{\alpha}_1=\check{\boldsymbol{\alpha}}_1\cdot\boldsymbol{x}$. The form of (2.31) then suggests that (at the classical level) any theory is unchanged under a transformation in the Gauss–Bonnet sense $\boldsymbol{x}\to\boldsymbol{x}+\check{\alpha}_{\mathrm{GB}}\boldsymbol{L}$ where $\boldsymbol{L}\equiv\check{\boldsymbol{\alpha}}_1-4\check{\boldsymbol{\alpha}}_3+2\check{\boldsymbol{\alpha}}_6$.

The quadratic Riemann–Cartan sector thus has a five-dimensional parameter space in general. When we demand homogeneity and isotropy as with cosmology, we might reasonably expect this number to be reduced. To identify the reduced D.o.F we should turn to the E.o.M. Doing so, we find the cosmological conditions eliminate a further two D.o.F from the quadratic Riemann–Cartan sector. Let us define two coordinates $\chi_1\equiv\frac{3}{2}\check{\alpha}_1+\frac{1}{4}\check{\alpha}_3-\frac{1}{4}\check{\alpha}_6$ and $\chi_2\equiv\frac{3}{2}\check{\alpha}_1+\frac{1}{2}\check{\alpha}_3+\frac{1}{4}\check{\alpha}_6$, which are oblivious to the Gauss–Bonnet content of the theory $\boldsymbol{\chi}_2\cdot\boldsymbol{L}=\boldsymbol{\chi}_1\cdot\boldsymbol{L}=0$. The cosmologically meaningful coordinates of the quadratic Riemann–Cartan sector are then equally oblivious, as we might expect, and are given by the three $\{\sigma_I\}$ parameters $\sigma_1\equiv\chi_1+\frac{1}{4}\check{\alpha}_2+\frac{1}{4}\check{\alpha}_5$, $\sigma_2\equiv\chi_2+\frac{1}{2}\check{\alpha}_2+\frac{3}{4}\check{\alpha}_4-\check{\alpha}_5$ and $\sigma_3\equiv\chi_2+\frac{1}{2}\check{\alpha}_2+\frac{1}{4}\check{\alpha}_4$.

We have already seen that the three $\{\check{\beta}_M\}$ of PGT^{q+} must reduce to two cosmic theory parameters for PGT torsion. Denoting these by $\{\upsilon_I\}$ we find $\upsilon_1\equiv\check{\beta}_1+3\check{\beta}_2$ and $\upsilon_2\equiv3\check{\beta}_3-\check{\beta}_1$. In eWGT^{q+} there *is* no $\check{\beta}_3$, but we find that its rôle is filled by ν, so that $\upsilon_2\equiv-\nu/6$.

We therefore find that the ten theory parameters of PGT^{q+} and eWGT^{q+} reduce to five cosmic theory parameters. The freedoms of the quadratic Riemann–Cartan sector are reduced from six to three, and those of the torsion and compensator sectors are reduced from three to two. We provide alternative forms of these parameters in Eqs. (B.24a) to (B.24d).

2.5.3 k-screening

Having defined the Lagrangian parameters relevant to cosmology, we are now in a position to express the E.o.M in a form valid simultaneously for *both* gauge theories. As before, these constitute a coupled system of four equations. For brevity, we write these in terms of dimensionless conformal time

$$\mathrm{d}\tau\equiv\mathrm{d}t/R, \tag{2.50}$$

and the dynamical variables introduced above, with the Friedmann gauge fixed:

$$\left(\delta\tilde{\mathcal{L}}_{\mathrm{T}}/\delta X\right)_{\mathrm{F}}\propto(\upsilon_2+\check{\alpha}_0)\,R\,(RX+\partial_\tau R)-8\kappa\sigma_3\partial_\tau^2X-4\kappa\sigma_1Y\partial_\tau Y$$

$$- 4\kappa X \left(\sigma_2 Y^2 - 4\sigma_3 \left(X^2 + k\right)\right), \tag{2.51a}$$

$$\left(\delta\tilde{\mathcal{L}}_{\mathrm{T}}/\delta Y\right)_{\mathrm{F}} \propto \left(4\upsilon_1 - \check{\alpha}_0\right) R^2 Y - 4\kappa \left(\sigma_3 - \sigma_2\right) \partial_\tau^2 Y + 16\kappa\sigma_1 Y \partial_\tau X + 4\kappa Y \left(\sigma_3 Y^2 - 4\kappa \left(\sigma_2 X^2 + \sigma_3 k\right)\right), \tag{2.51b}$$

$$\left(\delta\tilde{\mathcal{L}}_{\mathrm{T}}/\delta S\right)_{\mathrm{F}} \propto 12\upsilon_2 \partial_\tau^2 R + 12\left(\upsilon_2 + \check{\alpha}_0\right) R \left(\partial_\tau X - X^2\right) - 3\left(4\upsilon_1 - \check{\alpha}_0\right) R Y^2 - 12\check{\alpha}_0 k R + 2\kappa^{\frac{1}{2}} \varrho_{\mathrm{m}} + 8\Lambda R^3, \tag{2.51c}$$

$$\left(\delta\tilde{\mathcal{L}}_{\mathrm{T}}/\delta R\right)_{\mathrm{F}} \propto 12\upsilon_2 \left(2R\partial_\tau^2 R - \left(\partial_\tau R\right)^2\right) + 12\left(\upsilon_2 + \check{\alpha}_0\right) R^2 \left(2\partial_\tau X - X^2\right) - 3\left(4\upsilon_1 - \check{\alpha}_0\right) R^2 Y^2 - 12\check{\alpha}_0 k R^2 + 6\kappa\sigma_3 \big(16X^2 \left(X^2 + 2k\right) + Y^2 \left(Y^2 - 8k\right) + 16k^2 - 2(\partial_\tau Y)^2 - 16(\partial_\tau X)^2\big) + 12\kappa\sigma_2 \left((\partial_\tau Y)^2 - 2X^2 Y^2\right) - 4\kappa\varrho_{\mathrm{r}} + 12\Lambda R^4. \tag{2.51d}$$

A cursory examination of this system reveals a degree of similarity between the torsion equations (2.51a) and (2.51b) which we will mention again in Appendix B.4, along with the parameters σ_2 and σ_3, and υ_1 and υ_2. The single linear Riemann–Cartan parameter, $\check{\alpha}_0$, has an entirely different effect to the quadratic Riemann–Cartan parameters σ_1, σ_2 and σ_3, and while it mostly combines with υ_1 or υ_2, it couples uniquely with k in (2.51c). This gives us some insight into the cosmological overlap between Einstein–Hilbert and Yang–Mills theories: the latter are not expected to conventionally interact with the bulk curvature of space. In fact, a pure Yang–Mills theory

$$\check{\alpha}_0 = 0, \tag{2.52}$$

may be 'screened' from this curvature altogether, since the *single* parameter constraint

$$\sigma_3 = 0, \tag{2.53}$$

promptly eliminates k from the entire system. In the context of our opening remarks regarding ω_k in Section 2.1, this is a superficially disastrous choice of theory, in which the global geometry of space is decoupled from the dynamics. On the other hand, (2.53) is a tempting starting point for the study of $\mathrm{PGT}^{\mathrm{q}+}$ and $\mathrm{eWGT}^{\mathrm{q}+}$ cosmologies, since it eliminates many other unattractive derivative terms from the system, and does so with a very high degree of naturalness.

2.5.4 Cosmological normal scale-invariance

In our narrow ϕ-free definition of $\mathrm{PGT}^{\mathrm{q}+}$, the NSI condition on the gravitational sector (2.29) clearly imposes

$$\check{\alpha}_0 = \upsilon_1 = \upsilon_2 = 0. \tag{2.54}$$

The effect of (2.54) on Eqs. (2.51a) to (2.51d) is severely limiting, as it sets $\Omega_{\mathrm{m}} = \Omega_\Lambda = 0$ in all relevant solutions. We use this to write such theories off as *cosmologically* NSI. It should be noted that the cosmological NSI condition (2.54) is slightly less restrictive than (2.29). We also note that if ϕ were minimally included in $\mathrm{PGT}^{\mathrm{q}+}$ (i.e. without any term proportional to $\mathcal{D}_i\phi\mathcal{D}^i\phi$), from (2.49), the condition (2.30) would reduce to

$$\upsilon_2 = 0, \tag{2.55}$$

without any such loss of generality.

The select 33 critical cases of PGT^{q+} listed in Table 2.1 may now be categorised into 14 *cosmic classes* according to the effects of their defining parameter constraints on the general PGT^{q+} cosmology. This is illustrated in Fig. 2.1. Independent cosmic classes are labelled by letters, with a superscript denoting the minimum number of constraints that must be applied to the 'root' PGT^{q+} Lagrangian (2.26) to obtain them, e.g. Class ^{2}A, Class ^{4}L etc. Note that no critical case is completely determined by its cosmic class, in that there are always two or three non-cosmological constraints in the critical case definition which do not appear to affect Eqs. (2.51a) to (2.51d).

Undesirable as the NSI property is, we might expect to encounter it frequently if [152, 153] tend to suggest conformal field theories (CFTs) of gravity. CFTs are associated with a vanishing beta function in the renormalisation group flow, and so might be preferentially selected by the PCR criterion. We see from Fig. 2.1 that this is not actually a big problem. A heuristic explanation here is that the NSI and PCR criteria are sensitive to the nonlinear and linear theories, respectively. Thus, a theory may have a conformal symmetry which is broken in the nonlinear theory. Indeed, the theory (2) which we develop throughout the remainder of this chapter and in Chapter 3 is *not* NSI, but we will see in Chapter 5 that the SET of its linearisation around the simple vacuum assumed in [152, 153] has no trace, unless it acquires one by anomalous symmetry breaking. Much of the remainder of this thesis is spent showing how this is not actually a problem.

2.5.5 Motivated Cosmologies

Class ^{3}C: Einstein freezing

A key feature of Fig. 2.1 is that all critical cases begin with the Yang–Mills constraint (2.52). Beyond this, the k-screening condition (2.53) defines the most general vertex Class ^{2}A of the cube containing all critical cases with possible 2^{+} massless gravitons. To gain some traction, we will not start with Class ^{2}A, but enforce a third constraint on the torsion

$$\upsilon_1 = 0. \tag{2.56}$$

Class ^{3}C is the most general cosmology defined by these three constraints. A useful property common to Class ^{3}C and some of its children is that (2.51a) allows us to eliminate U from the system immediately

$$U = \frac{12\kappa Q\left(\left(\sigma_2 - \sigma_1\right) QH - \sigma_1 \partial_t Q\right)}{4\kappa\sigma_2 Q^2 - \upsilon_2}. \tag{2.57}$$

An energy balance equation may then be constructed by linear combination of (2.51c) and (2.51d)

$$\Omega_{\mathrm{r}} + \Omega_{\mathrm{m}} + \Omega_{\Lambda} + \Omega_{\Psi} + \Omega_{\Phi} = 0, \tag{2.58}$$

differing from (2.7) in the dependence of modified gravitational dimensionless energy densities Ω_{Ψ} and Ω_{Φ}, on the torsion. These are given in Appendix B.5, and are rational functions[15] of the form

$$\Omega_{\Phi} = \Omega_{\Phi}\big(\kappa^{\frac{1}{2}} Q \big| \sigma_1, \sigma_2, \upsilon_2\big), \quad \Omega_{\Psi} = \Omega_{\Psi}\big(\kappa^{\frac{1}{2}} \partial_t Q H^{-1}, \kappa^{\frac{1}{2}} Q \big| \sigma_1, \sigma_2, \upsilon_2\big). \tag{2.59}$$

[15] Note also that there is considerable freedom between these densities, if they are constrained only by (2.59), and that the notation is designed with Appendix B.3 and Class ^{4}H and Class ^{4}I in mind.

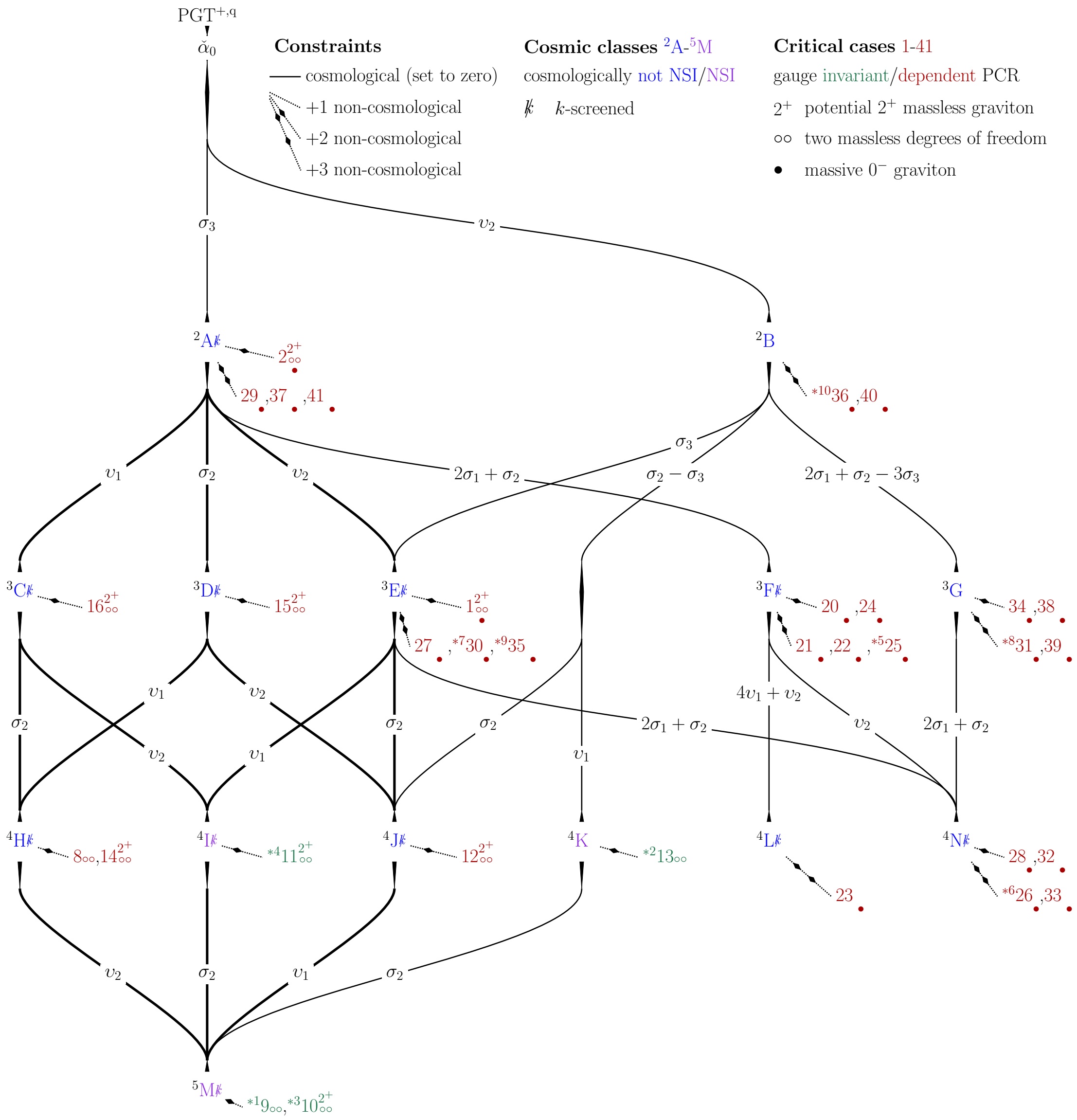


Fig. 2.1 The select 33 unitary, PCR critical cases of PGT^{q+} identified in [152, 153] and listed in Table 2.1 span 14 cosmic classes. Note that the traditional Einstein–Hilbert term is the first to be excluded, $\breve{\alpha}_0 = 0$. Desirable critical cases admit the possibility of a massless 2^+ graviton, i.e. Case 15, Case 16, Case 14, Case 12, Case ^{*4}11 and Case ^{*3}10. We cannot exclude Case 2 and Case 1 on the basis of their additional massive 0^- gravitons. Superficially, cosmic classes are excluded by cosmological NSI, which arises when $\breve{\alpha}_0 = \upsilon_1 = \upsilon_2 = 0$. By these criteria the only truly desirable cosmologies are clearly of Class ^{2}A, Class ^{3}C, Class ^{3}D, Class ^{3}E, Class ^{4}H or Class ^{4}J, and this restricts us to two faces of the cube at the far left of the diagram. All such cosmologies are k-screened, with $\breve{\alpha}_0 = \sigma_3 = 0$.

This dependence may in principle be eliminated in favour of H by means of the remaining torsion equation (2.51b) which takes the form

$$f_1 \frac{\partial_t^2 Q}{Q} + f_2 \frac{(\partial_t Q)^2}{Q^2} + f_3 \frac{\partial_t Q}{Q} H + f_4 \partial_t H + f_5 H^2 = 0, \tag{2.60}$$

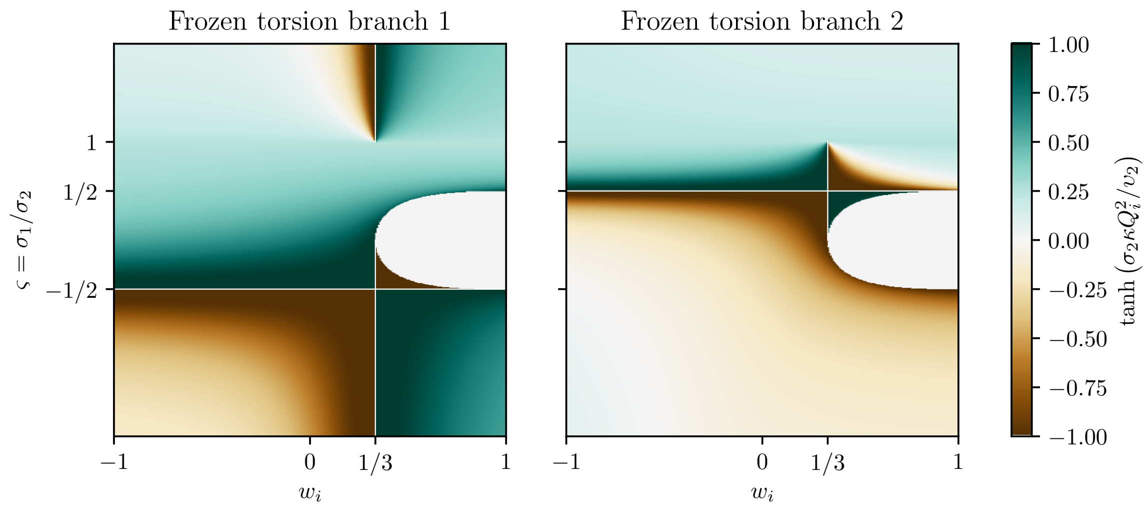


Fig. 2.2 Within Class ^{3}C, the root system (2.62) of constant torsion Q_i frozen out by a dominant cosmic fluid with E.o.S parameter w_i, depends on the ratio of cosmic theory parameters σ_1/σ_2. Freezing at a real torsion value generally appears possible for inflationary fluids ranging from dark energy $w_\Lambda \equiv -1$, through to curvature $w_k \equiv -1/3$ (although curvature cannot be re-imagined as a source in k-screened theories), and so on to matter $w_{\mathrm{m}} \equiv 0$. Radiation, at $w_{\mathrm{r}} \equiv 1/3$, clearly occupies a privileged position in the overall theory, while extension to 'stiff matter' with $w_{\mathrm{s}} \equiv 1$ may be impossible over a range of $\varsigma \equiv \sigma_1/\sigma_2$. Of particular interest is the case $\varsigma = 1$, which corresponds to Class ^{3}C*, and for which $\kappa Q_i^2 \equiv \kappa Q_{\mathrm{cor}}^2 \equiv \upsilon_2/4\sigma_1$ across all fluids except for radiation, which requires special treatment.

where the various coefficients are again confined to Appendix B.5 for the sake of brevity, and are also rational functions of the form $f_i = f_i\big(\kappa^{\frac{1}{2}}Q\big|\sigma_1,\sigma_2,\upsilon_2\big)$.

The coupled second-order system of (2.58) and (2.60) is generally challenging to solve, but despite the doubtful nature of the constraints (2.52) and (2.53), we are not disappointed if we look for the kind of curvature evolution suggested by GR. Since Class ^{3}C is fundamentally k-screened, it is logical to consider analogies with traditional $k = 0$ solutions – as discussed in Section 2.1, these are in contemporary focus anyway. The evolution of R in GR is often broken down into regimes where a particular cosmic fluid is dominant. For the material sources under consideration, (2.4a) and (2.4b) demand that R then approach a power-law in t, depending on the dominant E.o.S parameter w_i in (2.6)

$$H_{\mathrm{m}} \equiv 2/3t, \quad H_{\mathrm{r}} \equiv 1/2t, \quad H_\Lambda \equiv \sqrt{\Lambda/3}. \tag{2.61}$$

Remarkably, Class ^{3}C can mimic this behaviour. We require only that the modified gravitational densities be constant when a fluid of particular w_i is dominant $\Omega_\Phi + \Omega_\Psi = -1/g_i$, at which point (2.58) will then coincide with (2.7) up to a modified Einstein constant $\breve{\kappa} \equiv g_i\kappa$.

Examination of (2.59) suggests that this can be achieved by constant $Q = Q_i$, which in turn greatly simplifies (2.60) to a form which, for $H = H_i$ as in (2.61), *remains consistent* for as long as pure fluid dominance holds. We may thus hypothesise that a Universe of Class ^{3}C will routinely 'freeze out' into epochs of traditional flat GR behaviour. In this case the full complexity of the modified cosmological equations is confined to turnover epochs, and otherwise manifest in the specific value of the constant torsion Q_i and modified Einstein constant $\breve{\kappa}_i$ during pure fluid dominance.

The potential for this behaviour is worth some general investigation within Class ^{3}C, whose Lagrangian freedoms are partially parametrised by the ratio $\varsigma \equiv \sigma_1/\sigma_2$. Setting $Q = Q_i$ under a dominant cosmic fluid with E.o.S parameter $w = w_i$, the remaining torsion equation (2.60) may be solved for Q_i by setting $H_i = 2/3(1 + w_i)t$, which yields the following

$$\begin{aligned}(4\sigma_2/\upsilon_2)(12\varsigma^2 w_i - 4\varsigma^2 - 3w_i + 1)\kappa Q_i^2 &= 6w_i\varsigma^2 + 2\varsigma^2 + 6w_i\varsigma - 6\varsigma - 3w_i + 1 \\ &\pm 2\sqrt{9\varsigma^4 w_i^2 + 6\varsigma^4 w_i - 18\varsigma^3 w_i^2 + \varsigma^4 - 12\varsigma^3 w_i + 9\varsigma^2 w_i^2 - 2\varsigma^3 + 3\varsigma^2 + 12\varsigma w_i - 4\varsigma - 6w_i + 2}.\end{aligned} \tag{2.62}$$

The somewhat complementary branches of this root system are illustrated in Fig. 2.2. Superficially, this suggests that Einstein freezing can occur across many instances of Class ^{3}C for a variety of source fluids. Note however that radiation with $w_{\mathrm{r}} \equiv 1/3$ appears to occupy a special place in Class ^{3}C.

Numerically, it proves easy to induce such emergent flat GR behaviour, and this is best demonstrated by means of a series expansion out of the classical radiation-dominated Big Bang. When propagating the cosmological E.o.M, a convenient choice of dimensionless time similar to (2.50) is given by normalising with the contemporary Hubble number $\mathrm{d}\tilde{\tau} \equiv R_0 H_0 \mathrm{d}t/R$. When combined with the dimensionless scale factor $a \equiv R/R_0$, this has the advantage that the Friedmann equations of GR, (2.4a) and (2.4b), in the flat case become

$$(\partial_{\tilde{\tau}} a)^2 = \Omega_{\mathrm{r},0} + \Omega_{\mathrm{m},0} a + \Omega_{\Lambda,0} a^4, \quad (\partial_{\tilde{\tau}} a)^2 - a\partial_{\tilde{\tau}}^2 a = \Omega_{\mathrm{r},0} + \frac{1}{2}\Omega_{\mathrm{m},0} a - \Omega_{\Lambda,0} a^4, \tag{2.63}$$

i.e. a form where the contemporary dimensionless densities are the only free parameters. It is then easy to obtain the following power series for GR out of radiation dominance

$$a = \sqrt{\Omega_{\mathrm{r},0}}\tilde{\tau} + \frac{\Omega_{\mathrm{m},0}}{4}\tilde{\tau}^2 + \frac{\Omega_{\Lambda,0}}{10}\Omega_{\mathrm{r},0}{}^{\frac{3}{2}}\tilde{\tau}^5 + \mathcal{O}(\tilde{\tau}^6). \tag{2.64}$$

Applying this approach to Class ^{3}C results in a power series for a and separate series for Q and U. These are all rather cumbersome, but can be used to integrate the modified cosmological equations as follows. Assuming (2.57) remains valid, we can propagate the coupled second-order system in Q and R formed from the modified deceleration equation (the linear combination of (2.51c) and (2.51d) orthogonal to (2.58)), and (2.60), using (2.58) as a constraint. The resulting evolution of the comoving Hubble horizon H_0/aH is plotted against the scale factor a in Fig. 2.3, over a range of ς. Note that in Fig. 2.3, the initial conditions are tweaked to agree with the flat GR model as far as possible. This involves, for every instance of Class ^{3}C defined by ς, adapting υ_2 so that $\breve{\kappa} = \kappa$. We see that for ς of order unity, the radiation, matter and dark energy dominated regimes familiar from flat GR are cleanly picked out. The freezing of torsion by radiation, matter and dark energy is also apparent for some values of ς in Fig. 2.3.

Class ^{3}C*: dark radiation

From the analysis in Figs. 2.2 and 2.3 of the variable ς which parameterises Class ^{3}C, we see that an algebraically natural choice of theory defined by the additional constraint

$$\sigma_1 - \sigma_2 = 0, \tag{2.65}$$

or $\varsigma = 1$, is especially significant. We will refer to Class ^{3}C in combination with (2.65) as Class ^{3}C*. Since it is not defined by any critical case, Class ^{3}C* does not appear in the map of cosmologies in Fig. 2.1 – note however that Class ^{3}C* and Case 16 remain compatible.

To see the significance of (2.65), first note from Fig. 2.3 that Class ^{3}C* is defined by precisely the value $\varsigma = 1$ that imitates the expansion of flat GR cosmology, when propagated from the same initial conditions. In this case the Q_i and g_i all coincide at the same 'correspondence values' across the three w_i of radiation, matter and dark energy

$$\kappa Q_i^2 \equiv \kappa Q_{\rm cor}^2 \equiv \upsilon_2/4\sigma_1, \quad g_i \equiv g_{\rm cor} \equiv -4/3\upsilon_2, \tag{2.66}$$

and moreover do not deviate from these values during turnover epochs[16]. In order to recover the correct sign of the modified Einstein constant, we will need

$$\upsilon_2 < 0, \tag{2.67}$$

and likewise for real torsion

$$\sigma_1 = \sigma_2 < 0. \tag{2.68}$$

Confirmation of this behaviour can be seen in Fig. 2.2, since $\varsigma = 1$ is actually a contour in both branches of the frozen torsion value, except at the intersection with $w_i = 1/3$. Moreover, we see that $\varsigma = 1$ is one of the special cases of Class ^{3}C for which frozen torsion cannot escape the vertical radiation asymptote simply by switching branches. We refer to the solution (2.66) to Class ^{3}C*, in which flat GR evolution is naturally recovered, as the *correspondence solution* (CS).

While very encouraging in itself, in the absence of any measurement of Q_0 today and pinning $g_{\rm cor} = 1$ to recover $\breve{\kappa} \equiv \kappa$, the CS introduces no new parameters to cosmology: we thus seek to relax it. To do so, we will turn back to the series expansion out of the radiation-dominated Big Bang. It proves useful to define the dimensionless deviation from the correspondence torsion as $\varpi \equiv Q/Q_{\rm cor}$. Guided by Fig. 2.2, closer examination of the intersection of $w_i = 1/3$ with $\varsigma = 1$ reveals something interesting: the spectrum of possible $Q_{\rm r}$ or $\varpi_{\rm r}$ is in fact continuous here, introducing a free parameter. If therefore, we do not need to fix $\varpi_{\rm r} = 1$ at the singularity, the general power series for the scale factor in Class ^{3}C* is

$$\begin{aligned} a = {} & \frac{g_{\rm cor}}{\varpi_{\rm r}}\sqrt{\Omega_{\rm r,0}}\tilde{\tau} + \frac{\Omega_{\rm m,0}\left(3\,{\varpi_{\rm r}}^2+1\right){g_{\rm cor}}^2}{16\,{\varpi_{\rm r}}^2}\tilde{\tau}^2 + \frac{5\,{\Omega_{\rm m,0}}^2{g_{\rm cor}}^3\left({\varpi_{\rm r}}^2-1\right)}{512\,{\varpi_{\rm r}}^3}\frac{1}{\sqrt{\Omega_{\rm r,0}}}\tilde{\tau}^3 \\ & + \frac{{\Omega_{\rm m,0}}^3\left(27\,{\varpi_{\rm r}}^2-121\right){g_{\rm cor}}^4\left({\varpi_{\rm r}}^2-1\right)}{49152\,{\varpi_{\rm r}}^4\Omega_{\rm r,0}}\tilde{\tau}^4 \\ & + \frac{\left(-441\,{\varpi_{\rm r}}^4{\Omega_{\rm m,0}}^4 + 98304\,{\varpi_{\rm r}}^2\Omega_{\Lambda,0}\,{\Omega_{\rm r,0}}^3 + 1421\,{\varpi_{\rm r}}^2{\Omega_{\rm m,0}}^4 + 32768\,\Omega_{\Lambda,0}\,{\Omega_{\rm r,0}}^3 - 980\,{\Omega_{\rm m,0}}^4\right){g_{\rm cor}}^5}{1310720\,{\varpi_{\rm r}}^5} \\ & \times {\Omega_{\rm r,0}}^{-\frac{3}{2}}\tilde{\tau}^5 + \mathcal{O}(\tilde{\tau}^6), \end{aligned} \tag{2.69}$$

and by comparing (2.64) to (2.69) we see that the two series can be made to coincide by setting $\varpi_{\rm r} = 1$. Doing so guarantees the other half of the CS – the constancy of $\varpi = 1$ throughout the evolution – which

[16] It is important to note that the particular form of the E.o.M (2.58), (2.60) and particularly (2.57) only allow for this solution if a careful limit is taken.

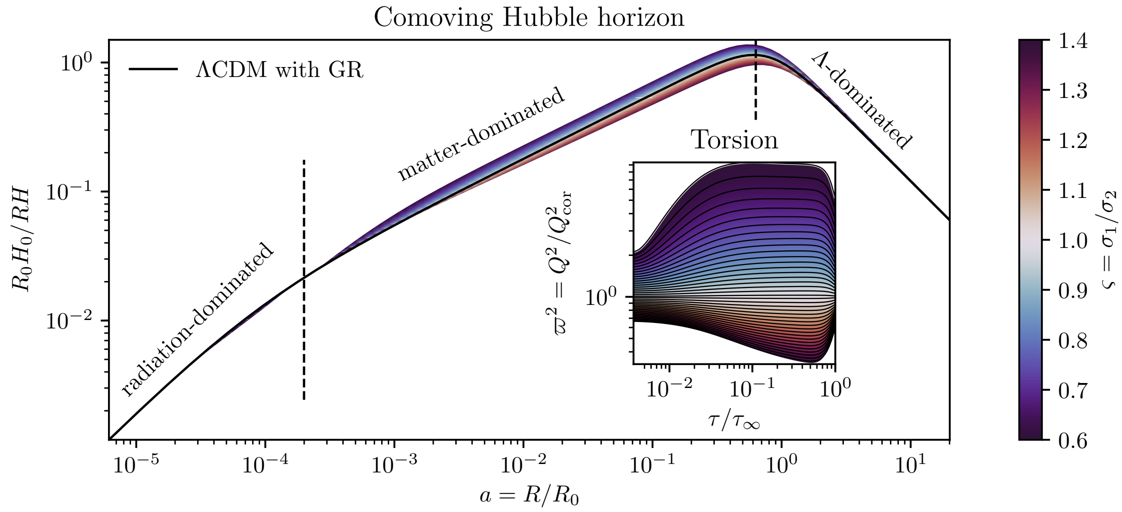


Fig. 2.3 *Main*: The cosmological equations of Class ^{3}C are propagated from $z \approx 1.63 \times 10^5$ (12 *e*-folds) using the corresponding primordial density parameters of flat GR (based on $\Omega_{\rm r,0} = 2.47 \times 10^{-5}$, $\Omega_{\rm m,0} = 0.3089 - \Omega_{\rm r,0}/2$ and $\Omega_{\Lambda,0} = 0.6911 - \Omega_{\rm r,0}/2$ with neutrinos neglected), with the GR evolution also shown. At this initial radiation-dominated epoch, $\breve{\kappa} = \kappa$ is fixed with $\upsilon_2 = 4\sigma_1/(\sigma_2 - 4\sigma_1)$. *Inset*: The Q torsion remains finite for the whole evolution, and may be plotted up to the Future Conformal Boundary at τ_∞. For general ς, each epoch of equality triggers a smooth transition to a new torsion value, the intermediate $Q_{\rm m}$ plateau is visible for $\varsigma < 1$. Arbitrarily close agreement with GR is seen as $\varsigma \equiv \sigma_1/\sigma_2 \to 1$, which corresponds to Class ^{3}C*. In this case, the CS keeps the torsion fixed throughout at $Q = Q_{\rm cor}$, or $\varpi \equiv Q/Q_{\rm cor} = 1$.

can be seen by examining the Class ^{3}C* power series for ϖ

$$\begin{aligned}\varpi =\ & \varpi_{\rm r} + \frac{3\,\Omega_{\rm m,0}\,g_{\rm cor}\left({\varpi_{\rm r}}^2 - 1\right)}{16}\frac{1}{\sqrt{\Omega_{\rm r,0}}}\tilde{\tau} + \frac{{\Omega_{\rm m,0}}^2{g_{\rm cor}}^2\left(18\,{\varpi_{\rm r}}^2 + 13\right)\left({\varpi_{\rm r}}^2 - 1\right)}{512\,\Omega_{\rm r,0}\,\varpi_{\rm r}}\tilde{\tau}^2\\ &+\frac{{\Omega_{\rm m,0}}^3{g_{\rm cor}}^3\left(324\,{\varpi_{\rm r}}^4 + 279\,{\varpi_{\rm r}}^2 + 299\right)\left({\varpi_{\rm r}}^2 - 1\right)}{49152\,{\varpi_{\rm r}}^2}{\Omega_{\rm r,0}}^{-\frac{3}{2}}\tilde{\tau}^3\\ &-\frac{{g_{\rm cor}}^4\left(-1620\,{\Omega_{\rm m,0}}^4{\varpi_{\rm r}}^6 - 1620\,{\varpi_{\rm r}}^4{\Omega_{\rm m,0}}^4 - 1462\,{\varpi_{\rm r}}^2{\Omega_{\rm m,0}}^4 + 98304\,\Omega_{\Lambda,0}\,{\Omega_{\rm r,0}}^3 - 2327\,{\Omega_{\rm m,0}}^4\right)\left({\varpi_{\rm r}}^2 - 1\right)}{1310720\,{\Omega_{\rm r,0}}^2{\varpi_{\rm r}}^3}\\ &\times\tilde{\tau}^4 + \mathcal{O}(\tilde{\tau}^5).\end{aligned} \tag{2.70}$$

This translates into precisely the relaxation of the CS we had sought. Rather than interpreting the effect of arbitrary $Q_{\rm r}$ through a time-varying renormalisation of the Einstein constant $\breve{\kappa}$, it is useful to cast it as a gravitational *extra component* which must be added to the bare (physical) matter in (2.4a) to account for the actual curvature evolution. This we will now do, and take the opportunity to combine the analysis with a crude stability check of the CS itself. To this end, we perturb the cosmological equations around the CS of some pure bare matter w_i, taking the origin of $\tilde{\tau}$ to be either the Big Bang as exited to the right or Future Conformal Boundary as approached from the left, i.e. $\mathrm{sgn}(3w_i + 1) = \mathrm{sgn}(\tilde{\tau})$. The perturbation of the correspondence curvature evolution is supposedly generated by a perturbation from

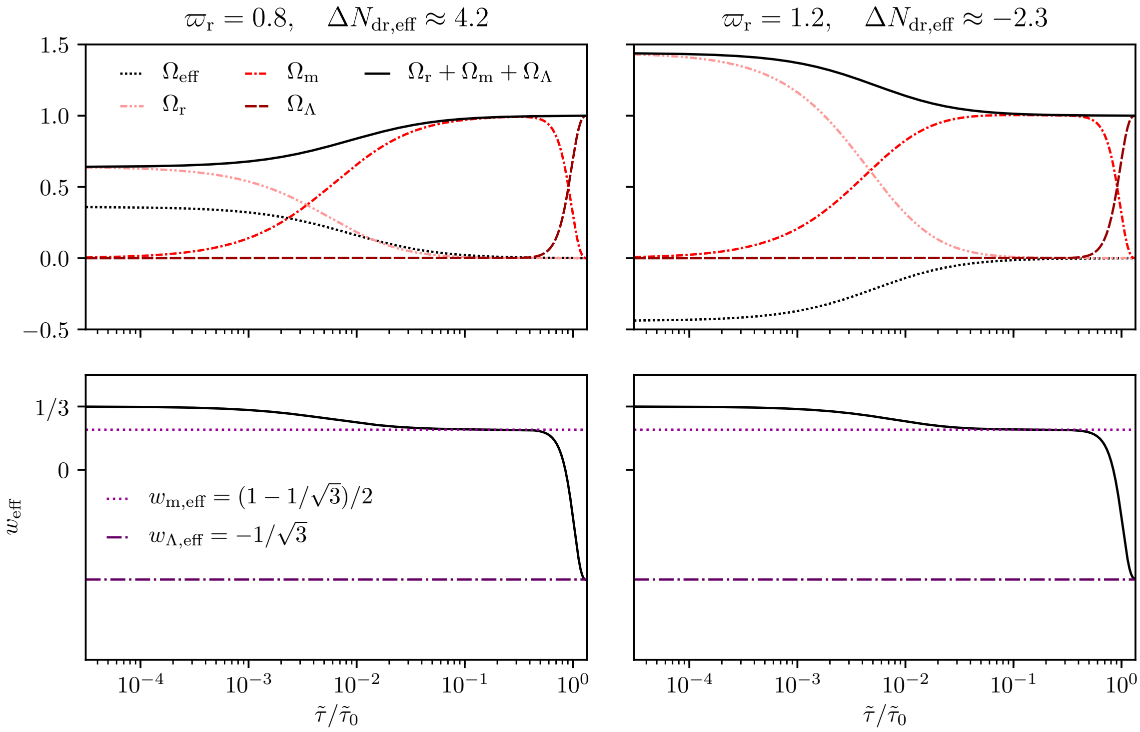


Fig. 2.4 Within Class ^{3}C*, density parameters and effective E.o.S parameter for a Big Bang with positive and negative dark radiation fractions. The effective E.o.S cleanly picks out the frozen regimes in (2.72), and consequently the dark sector redshifts away more slowly than radiation after the first turnover. Note that dark radiation with positive energy has a tendency to advance the epoch of equality.

correspondence torsion, or taking ε to be some small parameter

$$\varpi = 1 + \varepsilon\delta\varpi + \mathcal{O}(\varepsilon^2), \quad a = ((3w_i + 1)\tilde{\tau}/2)^{\frac{2}{3w_i+1}} + \varepsilon\delta a + \mathcal{O}(\varepsilon^2). \tag{2.71}$$

The solutions (2.71) can also be used to account for the extra components to which they correspond according to $(\partial_{\tilde{\tau}} a)^2 - a^{1-3w_i} = \varepsilon\kappa a^4\delta\rho/3H_0^2 + \mathcal{O}(\varepsilon^2)$. For the bare fluids anticipated here, we find that to first perturbative order the deviation from correspondence torsion typically decays away as a power law in normalised conformal time $\tilde{\tau}$ away from the Big Bang or towards the Future Conformal Boundary with the following forms[17]

$$\delta\varpi = \begin{cases} \left[c_1\tilde{\tau}^{-1} + c_2\right]^2 \\ \left[c_1\tilde{\tau}^{-\frac{3+\sqrt{3}}{2}} + c_2\tilde{\tau}^{-\frac{3-\sqrt{3}}{2}}\right]^2 \\ \left[c_1\tilde{\tau}^{\frac{3+\sqrt{3}}{2}} + c_2\tilde{\tau}^{\frac{3-\sqrt{3}}{2}}\right]^2 \end{cases} \quad a^4\delta\rho = \begin{cases} c_3 + c_4 a^{-2} & w_i = 1/3 \\ c_3 a^{-\frac{1+\sqrt{3}}{2}} + c_4 a^{-\frac{1-\sqrt{3}}{2}} & w_i = 0 \\ c_3 a^{1+\sqrt{3}} + c_4 a^{1-\sqrt{3}} & w_i = -1. \end{cases} \tag{2.72}$$

We take this to confirm the stability of the CS under pure fluid dominance. The obvious exception is the arbitrary constant torsion deviation under bare *radiation* dominance. This was of course anticipated as part of the relaxation procedure, and it need not be perturbative at all. Note that both parts

[17] The precise dependece of c_3 and c_4 on c_1 and c_2 is suppressed for brevity.

of (2.72) are consistent in that a *decaying* deviation from correspondence torsion is manifest as a strictly sub-dominant extra component. After a while, the extra component may be approximated by the contribution from the slowest-decaying torsion mode, and we see that it quietly redshifts away under the dominant bare matter in all cases but bare radiation. For this reason, we anticipate an arbitrary co-dominant dark radiation component to accompany bare radiation until the epoch of equality, a small amount of hot dark matter with $w_{\mathrm{m,eff}} \approx 0.211$ to accompany bare matter and, after the contemporary turnover, a miniscule amount of non-phantom dark energy with $w_{\Lambda,\mathrm{eff}} \approx -0.577$ to accompany bare dark energy. These values, which we introduced in (2.9) in Section 2.1, can readily be obtained from (2.72), and we will find a general formula for them in Chapter 3.

Numerical investigation suggests that this version of events is surprisingly robust, in that large positive or negative dark radiation fractions in the early Universe are typically eliminated by the first turnover they encounter. The analytic predictions for the effective E.o.S parameter are borne out in Fig. 2.4. The ability of the theory to recover LCDM evolution at late times over a wide range of ϖ_{r} is especially striking in toy Universes without bare matter, as illustrated in Fig. 2.5: the CS superficially resembles a damped harmonic attractor out of initial dark radiation dominance[18].

In the broadest terms, we can understand the arbitrary-ϖ_{r} solution to Class $^3\mathrm{C}^*$ as a positive or negative dark radiation component in the early Universe. A crude translation into the nomenclature of LCDM mentioned in Section 2.1 is simply to absorb this dark radiation into the effective BSM relativistic D.o.F $\Delta N_{\mathrm{dr,eff}}$ as follows

$$\Delta N_{\mathrm{dr,eff}} = \left(\varpi_{\mathrm{r}}^{-2} - 1\right)\left(\frac{8}{7}\left(\frac{11}{4}\right)^{4/3} + N_{\nu,\mathrm{eff}}\right). \tag{2.73}$$

This heuristic formula is the basis of the $\Delta N_{\mathrm{dr,eff}}$ values referenced in Fig. 2.4 and Fig. 2.5, given the Planck 2018 estimate of $N_{\nu,\mathrm{eff}} = 2.99 \pm 0.17$ [24]. This estimate may fall foul of circularity arguments due to the GR interpretation of the Planck data, and direct $\Delta N_{\nu,\mathrm{eff}}$ estimations [167] based on BBN may be more appropriate. Finally we emphasise that the dark radiation approximation *remains* an approximation: the general arbitrary-ϖ_{r} solution predicts a complicated dark sector with a dynamical E.o.S.

2.6 Closing remarks

Before summarising our results, we note that this chapter is principally directed at the PGT. However, the classical equivalence of $\mathrm{PGT}^{\mathrm{q}+}$ and $\mathrm{eWGT}^{\mathrm{q}+}$ cosmologies should save considerable time as the latter field develops. We are hopeful that it may also be generalised to other simple spacetimes, such as pp-waves, anisotropic Bianchi models and axisymmetric sources. Certain caveats regarding the guiding references [152, 153] should also be reiterated: these represent preliminary investigations into the $\mathrm{PGT}^{\mathrm{q}+}$ because they only extend to the linear theory. Moreover, we do not necessarily expect them to extend to $\mathrm{eWGT}^{\mathrm{q}+}$ at any level of approximation. We note that work has since been undertaken [158] to perform a similar systematic search for unitary PCR instances of $\mathrm{WGT}^{\mathrm{q}+}$ with the ultimate aim of a full $\mathrm{eWGT}^{\mathrm{q}+}$ survey. Next, the additional gauge symmetries which define the various critical cases have not themselves been studied: there is no guarantee that they survive in the nonlinear theory. Of greater concern is the question of renormalisability, as the power-counting formalism is very much a *first step* in its determination. The need for a nonlinear quantum feasibility analysis is thus obvious. One

[18] We will not attempt to prove that the critical solution is actually an attractor state, the Hessian analysis and dynamical systems approach will be constructed in Chapter 3.

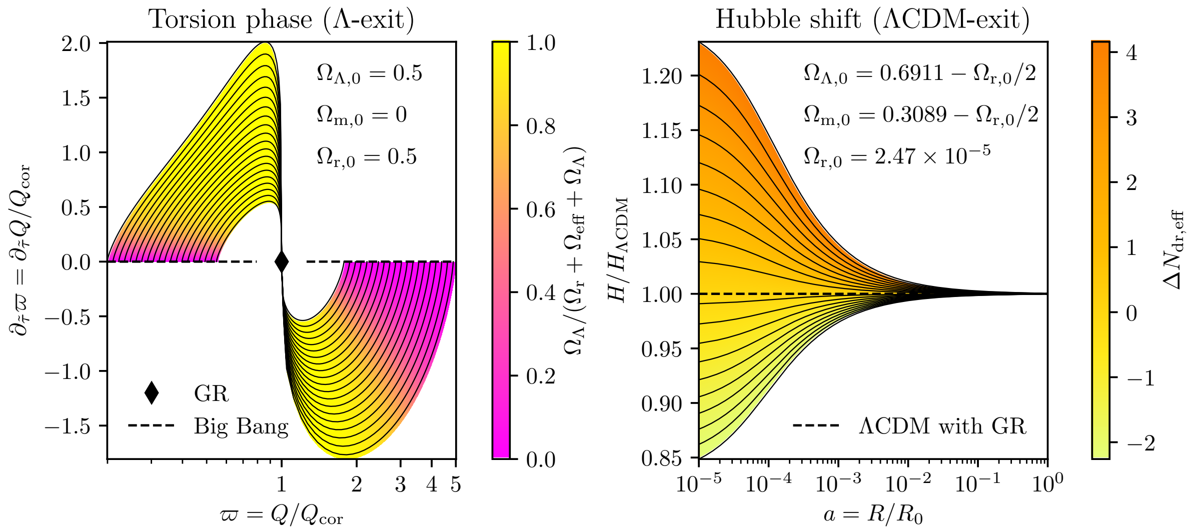


Fig. 2.5 Reliable emergence of Einsteinian cosmology from Class ^{3}C*. *Left*: The CS attracts the torsion to the value Q_{cor}, here illustrated in the phase space of Q for a toy model with no matter. Torsion is at rest at the Big Bang when the only sources are dark energy and radiation as shown here – though if matter is present it begins to decay immediately, and propagates off a parabola in phase space; *right*: compared to LCDM, initial dark radiation allows one-parameter tuning of the expansion rate during radiation-dominance. Compared to the equivalent plot in [115] we are allowed both increased and decreased early expansion because our extra component is effective, furthermore the effect is heavily suppressed at modern times.

possible method is the Hamiltonian analysis [168, 169], which was used to eliminate certain of Sezgin and Nieuwenhuizen's theories [156] on the grounds of constraint bifurcation and field activation. We will perform this investigation in Chapter 4, and address the problems it reveals in Chapter 5; for now, we return to our summary.

Within PGT^{q+}, we grouped 33 of the 58 new critical cases into 14 cosmic classes. Most of these classes are k-screened, in the sense that the evolution of the Universe is decoupled from the spatial curvature. We stress that this does not equate to an assertion that $k = 0$, but rather that the flat, open or closed nature of the geometry does not affect the expansion rate or torsion evolution. Similar effects might be expected in metric-based theories based on the Weyl tensor [57], however we show in Appendix A.8 that k-screened theories *may be constructed independently of the quadratic invariants of the PGT analogue of the Weyl*. Screened theories include Class ^{3}C which contains Case 16.

It also includes the special case Class ^{4}H, which contains Case 14: this theory will be tangential to future chapters and so we discuss it fully in Appendix B.3, mentioning only the main results here. We also discuss in Appendix B.3 the cosmological limitations following from the unitarity conditions in Table 2.1, noting that these do not affect the conclusions drawn in this chapter. Despite k-screening, Class ^{3}C and Class ^{4}H can be understood to mimic the cosmology of GR, powered 'under the hood' by involved curvature-torsion interactions. In Class ^{3}C, flat GR cosmology emerges through 'Einstein freezing', when a pure fluid with E.o.S parameter w_i becomes dominant, up to a w_i-specific renormalisation of the Einstein constant that depends on a parameter of the theory ς. Such a renormalisation is better understood in terms of an extra-component model, in which context it could be exploited for various purposes, such as dark energy enhancement – this is of course objectionable on the grounds of fine-tuning.

To eliminate ς *naturally* we may either change the quantum theory to the Case 14 of Class ^{4}H, or pick an instance of Class ^{3}C that appeals on classical and algebraic grounds without contradicting Case 16, such as Class ^{3}C*. Class ^{4}H requires $\varsigma \to \infty$ in our notation, but remains a promising theory in that the Friedmann equations emerge *exactly along with an effective* $k \leq 0$. Class ^{3}C* sets $\varsigma = 1$, but the CS can be found in which the effective $k = 0$.

In thus avoiding fine-tuning, we have in some sense returned to spatially flat GR cosmology. Remarkably however, the special significance of radiation in Class ^{3}C gives rise to an extra torsion freedom at the radiation-dominated Big Bang in Class ^{3}C*, and this allows the complexity of the theory to shine through. In the extra-component picture, this is manifest as a dark 'tracker matter' fraction, whose E.o.S reflects that of the dominant cosmic fluid. Post-equality, this matter is always subdominant, and its principal effect is that of dark radiation in the early Universe.

We have been driving at a popular proposal in the resolution of the H_0 discrepancy, which is worth some explanation. Generally, the expansion history of the Universe must be tweaked so as to revise the CMB-inferred value of H_0 and h *upwards*, towards less history-sensitive measurements (e.g. from the SH0ES program or HOLiCOW project). The CMB power spectrum can be roughly characterised by two quantities [117, 170, 118, 171], the *shift parameter* $\mathcal{R}$ and multipole position l_{a} of the first peak

$$\mathcal{R} \equiv \mathsf{H}\sqrt{\omega_{\mathrm{m}}} D_{\mathrm{A}}(z_{\mathrm{rec}}), \quad l_{\mathrm{a}} \equiv \pi \frac{D_{\mathrm{A}}(z_{\mathrm{rec}})}{r_{\mathrm{s}}}. \tag{2.74}$$

These quantities rely on the comoving angular diameter distance to recombination (as a proxy for CMB decoupling), D_{A} at z_{rec}, and sound horizon r_{s} at that same epoch t_{rec} – both model dependent scales. Expressions for D_{A} which hold for general k illustrate its sensitivity to the expansion history

$$D_{\mathrm{A}}(z_{\mathrm{rec}}) = (1 + z_{\mathrm{rec}}) d_{\mathrm{A}}(z_{\mathrm{rec}}) = \frac{\sin\left(\sqrt{-\Omega_{k,0}} \int_0^{z_{\mathrm{rec}}} \frac{H_0 \mathrm{d}z}{H}\right)}{H_0 \sqrt{-\Omega_{k,0}}} = \frac{\sinh\left(\sqrt{\omega_k} \int_0^{z_{\mathrm{rec}}} \frac{\mathsf{H} \mathrm{d}z}{H}\right)}{\mathsf{H}\sqrt{\omega_k}}, \tag{2.75}$$

while r_{s} depends on both the expansion history and photon-baryon sound speed[19] c_{s}

$$r_{\mathrm{s}} = \int_0^{t_{\mathrm{rec}}} \frac{c_{\mathrm{s}} \mathrm{d}t}{a}, \quad c_{\mathrm{s}} = \frac{1}{\sqrt{3\left(1 + 3\omega_{\mathrm{b}} a / 4\omega_{\mathrm{r}}\right)}}. \tag{2.76}$$

If z_{rec} is held constant, a general increase in H for $z < z_{\mathrm{rec}}$ consistent with local observations will reduce D_{A} as expressed in (2.75). In order to preserve l_{a} in (2.74), we will therefore need a decrease in r_{s}. This can in turn be achieved by increasing H for $z_{\mathrm{rec}} < z$ and thus reducing t_{rec} by (2.76). This mechanism is traditionally favoured because it impinges on relatively few of LCDM's moving parts. Of these parts, perhaps the strongest constraints come from BBN: if photons decouple at an earlier time then neutrinos decouple at a higher temperature. Fortunately, the implications for for the ratios of light nuclei are thought to be (just) consistent [117] with a tension-resolving tweak to the early expansion rate. On the other hand, recent work combining BBN and baryon acoustic oscillation (BAO) constraints (which probes only the background evolution so long as neutrino drag is neglected) indicates only a tension reduction to 2.6σ [119].

A selective increase in the early expansion rate independent of other density parameters is qualitatively implied by our model: the relaxed or arbitrary-ϖ_{r} soluton to Class ^{3}C*. Many alternative methods have

[19] Recall also that ω_{b} and ω_{r} in (2.76) are *contemporary* densities, according to (2.5).

been employed in recent years, including early dark energy [117], dark-sector interactions [172, 173, 121] or varying Λ models [118]. These tend to lie on a spectrum between data-driven searches and theoretically motivated proposals for an extra component. Such motivations arise, for example, in particle physics [115] and string theory [174], though they mostly bear fruit in the form of toy models. Our proposal has the advantage that the effect emerges from an *independently motivated* theory of gravity, and can be compared to (e.g.) similar applications of the ghost-free bimetric theory [117]. A more obvious approach is to simply introduce additional ultrarelativistic species such as sterile neutrinos and so to alter ΔN_{eff} – we stress again that the quantity $\Delta N_{\mathrm{dr,eff}}$ is introduced in Section 2.5.5 for convenience only, and does not confer any such ad hoc species. This is significant as some BBN-oriented studies [122] specifically assume thermal particles in equilibrium with the SM plasma, while the Rayleigh–Jeans tail of the CMB can constrain some dark electromagnetism models [120]. The term 'dark radiation' is also something of a misnomer, since our theory makes a clear prediction as to the evolution and present intensity of the pseudoscalar torsion mode, which ought to be nearly constant for $z \ll z_{\mathrm{rec}}$, and on the order of the Planck mass

$$Q_0 \sim m_{\mathrm{p}}. \tag{2.77}$$

As we observed earlier, this is precisely the torsion mode which is expected to interact with matter, introducing the potential for detection and falsifiability. On the other hand it must be noted that (2.77) relies on a somewhat naïve interpretation of $\mathrm{PGT}^{\mathrm{q}+}$ in which the $\{\alpha_I\}$ and $\{\beta_M\}$ along with the $\{\sigma_I\}$ and $\{\upsilon_I\}$ are assumed to be of order unity. There is reason to believe [175] that in $\mathrm{eWGT}^{\mathrm{q}+}$ any experiment would only be able to determine the quantitiy $\sigma_1 Q_0^2$, and that σ_1 need *not* be of order unity. In the context of $\mathrm{PGT}^{\mathrm{q},+}$, we will show in Chapter 5 that σ_1 may be identified with the coupling strength of a fourth-order correction to the Newtonian limit, suggesting that it might be constrained using parameterised post-Newtonian (PPN) methods without the need for torsion interactions with matter. It should moreover be noted that attempts at measuring torsion are generally specific to the theory, with most attention naturally granted to ECT. The series [124, 176] provides a current review of spin-gravity interaction in theory and practice. Some quite concrete proposals have been made [177] based on microstructured matter or nonminimal couplings of $\mathcal{T}^i{}_{jk}$ and $\mathcal{R}^i{}_{jkl}$ to the matter fields φ: no such couplings are assumed in ten-parameter $\mathrm{PGT}^{\mathrm{q}+}$.

Our classical results are also preliminary, since we have restricted our attention to *background* cosmology. Compared to GR, our gravity theory is not so much modified as completely rewritten, and its effect on perturbations will eventually require a dedicated study [119]. In the short term, we envisage only a small modification to a publicly available Markov chain Monte Carlo (MCMC) engine such as `COSMOMC` [178] or `CLASS` [179], using e.g. a spline approximation of the E.o.S parameter set out in (2.9) in Section 2.1. The same basic questions surround, for example, solar system tests: we will examine this matter in Chapter 5.

We have had nothing to say about inflation, dark matter or dark energy. We cannot dismiss the idea that k-screening may be of some relevance to the flatness problem, or that the general unpredictability of Class ^{3}C cosmology at turnover epochs may help explain the cosmic coincidence. At the classical level, Class ^{3}C* gravity only suggests a route out of the H_0 tension, and in this sense it is economical. Our model invokes a natural freedom early in the radiation-dominated epoch, which is eliminated by dark energy at the Future Conformal Boundary. This has the advantage of extending LCDM by only *one* parameter. The obvious *zero* parameter grail would be to replace the classical singularity with a torsion-driven de Sitter expansion in the primordial Universe which naturally exits to the correct dark radiation fraction. In the next chapter, we will instead focus on the de Sitter state at our end of the Universe. We will find that this state is also a natural product of our theory.

Chapter 3

The scalar-tensor analogue and emergent dark energy

Abridged from W. E. V. Barker, A. N. Lasenby, M. P. Hobson and W. J. Handley, Physical Review D **102**, 084002 (2020), arXiv:2006.03581 [gr-qc].

3.1 Introduction

The gauge-theoretic approach assumed in Chapters 1 and 2 is neither the most accessible nor the most minimal route to modifying GR. In order to bypass Lovelock's theorem, scalar-tensor theories couple various scalar fields[1] ϕ to the metric $g_{\mu\nu}$ on the *curved* (V_4) spacetime $\mathcal{M}$ [180]. This approach is prevalent in EFT extensions to GR, and even used to model inflation within LCDM [181]. Scalar-tensor theories are tractable and very widely studied, and in this sense they are *self-motivating*.

In Chapter 2 we used the homogeneity and isotropy of the SCP to partition a select 33 of the 58 novel cases into phenomenological classes. The Class ${}^{3}\mathrm{C}^{*}$ theory reproduces the LCDM background. Moreover, an early-time deviation from LCDM dilutes away as dark radiation, qualitatively suited to ease the present tension [46, 47] between CMB-inferred and locally-observed determinations of the contemporary Hubble number. The more general Class ${}^{2}\mathrm{A}^{*}$ has an additional massive 0^{-} D.o.F, but is hitherto unexplored. Separately, we recall that the cases underlying these classes simultaneously contain two massless (possibly 2^{+}) D.o.F and support the usual gravitational wave polarisations [182]. Notwithstanding our analysis in Chapter 2, the cosmological equations of $\mathrm{PGT}^{\mathrm{q},+}$ are quite cumbersome and opaque. This has led to fruitful, but often piecewise investigations for almost forty years (see e.g. [138, 141, 136, 148] or a review of the substantial literature [139]).

The first aim of this chapter is to develop a simple bi-scalar-tensor theory – the *metrical analogue* (MA) – which reproduces the spatially-flat background cosmology of $\mathrm{PGT}^{\mathrm{q},+}$. The general MA will be given in Eq. (3.11b) and provides a unified framework for future infrared (IR) investigation by the broader community. Since the MA is free of both torsion and quadratic curvature invariants, we find that it offers a refreshingly clear statement of the IR. Just as $h_i^{\ \mu}$ is in some sense the *square root* of $g_{\mu\nu}$, so the

[1] Note the galileon ϕ is not to be confused with the dilaton compensator introduced in (2.27).

MA contains a non-canonical kinetic term of the form $\sqrt{|X^{\phi\phi}|}$, where $X^{\phi\phi} \equiv \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. Such fields are known in cosmology as *Cuscutons* [183]: they provide a rich phenomenology [184], but are naturally challenging to motivate (see e.g. EFT applications in Hořava–Lifshitz gravity [185]). We will show that teleparallelism has an Einstein–Hilbert MA, while the MA of ECT theory is a pure *Cuscuton*.

The second aim of this chapter is to use the MA to study the IR of certain novel cases, which were partly motivated in the UV. We will show that Class ^{2}A* of PGT$^{q,+}$ inherits the dark radiation of Class ^{3}C*, while the 0^- mass generates *dark energy*. The *Cuscuton* tends to 'stall' the cosmology in a state equivalent to LCDM. With relevance to the Hubble tension and cosmological constant problem [30, 186], our results build the case for further careful scrutiny of the underlying novel cases.

The remainder of this chapter is set out as follows. In Sections 3.2.1 and 3.2.2 we map PGT$^{q,+}$ to the MA. In Section 3.3.2 we provide a brief recap on the novel theories of interest, and their potential for renormalisability. In Section 3.3.3 we show that Class ^{2}A* can undergo accelerated expansion in the presence of a *negative* bare cosmological constant. In Section 3.3.4 we find an alternative solution (the CS) in which LCDM is recovered, but the cosmological constant is provided entirely by the gravitational sector. Conclusions follow in Section 3.4.

3.2 Theoretical development

3.2.1 Metric theories

The generalised galileon, more commonly known as Horndeski theory [187], is the most general ϕ–$g_{\mu\nu}$ coupling with maximally second-order field equations. Avoidance of higher-order field equations is a simple (yet insufficient) precaution against ghosts given by Ostrogradsky's theorem [188]. The generalised bi-galileon [189] introduces a second scalar ψ and is known *not* to be the most general second-order bi-scalar-tensor theory [190], but follows a simple prescription and is also often called Horndeski theory. The generality of the bi-galileon is provided by six arbitrary G-functions. Of these, it suits our needs to discard G_3^ϕ, G_3^ψ, G_5^ϕ and G_5^ψ (adopting the usual notation [191]) for a total Lagrangian

$$L_{\mathrm{T}} = G_2(\phi,\psi|X^{\phi\phi},X^{\psi\psi}) + G_4(\phi,\psi)R + L_{\mathrm{M}}(\Phi|g). \tag{3.1}$$

Note that G_2 couples $\partial\phi$ and $\partial\psi$ to $g_{\mu\nu}$, and G_4 non-minimally couples ϕ and ψ to ∂g and $\partial^2 g$ via the Ricci scalar $R \equiv R^{\mu\nu}{}_{\mu\nu}$, where the Riemann tensor $R \sim \partial\Gamma + \Gamma^2$ is given in Eq. (1.7) and the Levi–Civita connection $\Gamma^\sigma_{\mu\nu}$ is of the form $\Gamma \sim g^{-1}\partial g$. As with GR, one cannot formally fit the whole SM into the matter Lagrangian $L_{\mathrm{M}}(\Phi|g)$. This is an elementary but occasionally overlooked limitation of metric theories: the matter fields Φ must be tensorial representations of $\mathrm{GL}(4,\mathbb{R})$, and are thus *bosonic*. Note also that while ϕ and ψ are historically termed *galileons*, the covariantisation of the theory with respect to $g_{\mu\nu}$ breaks the Galilean shift symmetry. In exchange, (3.1) acquires diffeomorphism invariance and (like GR) may be interpreted as a geometric $\mathbb{R}^{1,3}$ gauge theory.

Various other geometric gauge theories have been proposed, as we discovered in Chapter 2. Promotion of the proper, orthochronous Lorentz rotations to a local symmetry yields the Poincaré gauge theory (PGT) of $\mathbb{R}^{1,3} \rtimes \mathrm{SO}^+(1,3)$. The most general such theory (up to quadratic order in the field strengths and invariant under parity inversions) can be obtained by consolidating the various parts in Section 2.2.2, and is cast on $\check{\mathcal{M}}$ (i.e. Minkowski space M_4 rather than the Riemann–Cartan space U_4) as

$$\begin{aligned} L_{\text{T}} = &-\frac{1}{2}\hat{\alpha}_0 {m_{\text{p}}}^2\mathcal{R} + {m_{\text{p}}}^2\mathcal{T}_{ijk}\left(\beta_1\mathcal{T}^{ijk} + \beta_2\mathcal{T}^{jik}\right) + \beta_3 {m_{\text{p}}}^2\mathcal{T}_i\,\mathcal{T}^i + \alpha_1\mathcal{R}^2 + \mathcal{R}_{ij}\left(\alpha_2\mathcal{R}^{ij} + \alpha_3\mathcal{R}^{ji}\right) \\ &+ \mathcal{R}_{ijkl}\left(\alpha_4\mathcal{R}^{ijkl} + \alpha_5\mathcal{R}^{ikjl} + \alpha_6\mathcal{R}^{klij}\right) + L_{\text{M}}(\Phi,\Psi|h,A). \end{aligned} \tag{3.2}$$

This general theory is termed $\text{PGT}^{\text{q},+}$, and is parameterised by ten dimensionless coupling constants. Note that *fermionic* fields Ψ are now permitted[2] in $L_{\text{M}}(\Phi,\Psi|h,A)$ as representations of $\text{SL}(2,\mathbb{C})$, the spin group of M_4 which universally covers $\text{SO}^+(1,3)$. This is of vital importance: *all* the fundamental matter fields from the SM, the quarks and leptons with $J=1/2$, exist in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation[3] as Dirac spinors. Arbitrary composite states are also allowed, such as the three-quark Δ baryons which exist in the $(\frac{1}{2},\frac{1}{2})\otimes((\frac{1}{2},0)\oplus(0,\frac{1}{2}))$ representation as $J=3/2$ Rarita–Schwinger particles, and so on. With or without torsion, the provision of Lorentz indices in the gravitational sector is thus *crucial* for gravitational coupling to matter. We recall also from Section 2.2.2 the Maxwell-like terms in (3.2) are motivated by analogy to the Yang–Mills structure of the SM: since Eqs. (2.16) and (2.17) are at lower order than (1.7), maximally second-order field equations are guaranteed by construction.

3.2.2 Scale-invariance

Pushing the SM analogy further, one considers scale-invariance. This pertains to local conformal transformations

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \phi' = \Omega^{-1}\phi, \quad \psi' = \Omega^{-1}\psi, \tag{3.3a}$$

$$b'^i{}_\mu = \Omega b^i{}_\mu, \quad A'^{ij}{}_\mu = A^{ij}{}_\mu, \tag{3.3b}$$

or Weyl transformations $e^\rho = \Omega$ as given in (2.18). The Lagrangia (3.1) and (3.2) are scale-invariant if they transform with weight -4, which cancels with the measure $\sqrt{-g}$, or $h^{-1} \equiv \det b^i{}_\mu$. A scale-invariant $\text{PGT}^{\text{q},+}$ has $\alpha_0 = \beta_1 = \beta_2 = \beta_3 = 0$ (i.e. Eq. (2.29)) which eliminates the explicit mass scale m_{p}. By convention, ϕ and ψ have weight -1 [192] and $A^{ij}{}_\mu$ has weight 0 [92]. We recall as an aside that an inhomogeneously rescaling $A^{ij}{}_\mu$ was recently used in an *extension* of *Weyl* gauge theory (eWGT) [92]. Quite unlike PGT, eWGT is scale-invariant *by construction.* However, when expressed in terms of scale-invariant variables [193–195], the quadratic, parity-preserving version ($\text{eWGT}^{\text{q},+}$) was shown in Chapter 2 to be dynamically equivalent to $\text{PGT}^{\text{q},+}$ under the SCP. At this level, $\text{PGT}^{\text{q},+}$ and $\text{eWGT}^{\text{q},+}$ differ only through a scale-dependent interpretation of the coupling constants. We will briefly return to $\text{eWGT}^{\text{q},+}$ in closing.

3.2.3 The full metrical analogue

We will now construct an instance of (3.1) which mimics (3.2) under the spatially-flat SCP. Adopting dimensionful Cartesian coordinates on $\mathcal{M}$ and setting $k=0$ in (2.1), the FLRW metric has interval

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2\mathrm{d}\mathbf{x}^2. \tag{3.4}$$

[2] Note that the Φ and Ψ will partition the general φ in (2.13).

[3] We refer to the bracket convention of labelling irreps of $\text{SO}^+(1,3)$ as direct products of $\text{SU}(2)$ representations.

The dimensionless scale factor[4] a provides the Hubble number $H \equiv \partial_t a/a$, and our isotropic differential vector notation is as in (1.21). Under conformal transformations of the form (3.3a), the form of (3.4) is always preserved by implicit combination with the diffeomorphism

$$\mathrm{d}t' = \Omega^{-1}\mathrm{d}t, \quad H' = \Omega^{-1}(H - \partial_t\Omega). \tag{3.5}$$

Analogous Cartesian coordinates $\gamma_{\mu\nu} = \eta_{\mu\nu}$, assumed to transform according to (3.5) under Weyl rescalings of the form (3.3b), then allow us to equate component values $g^{\mu\nu} \stackrel{\text{an}}{=} \eta^{ij}h_i{}^\mu h_j{}^\nu$ and $g_{\mu\nu} \stackrel{\text{an}}{=} \eta_{ij}b^i{}_\mu b^j{}_\nu$. Our 'analogue equality' flags the notational abuse of incompatible tangent spaces. The torsion tensor on $\check{\mathcal{M}}$ is restricted by the SCP to the scalar U and pseudoscalar Q in (2.37), which are the 0^+ and 0^- sectors [129, 196, 197] $\mathcal{T}^i{}_{jk} = (\hat{\boldsymbol{e}}_t)^l(\frac{2}{3}U\delta^i_{[k}\eta_{j]l} - Q\epsilon^i{}_{ljk})$. These are homogeneous cosmological fields in the same sense as ϕ and ψ, inviting the analogue of torsion on $\mathcal{M}$

$$\phi \stackrel{\text{an}}{=} \frac{2}{3}U - 2H, \quad \psi \stackrel{\text{an}}{=} Q. \tag{3.6}$$

Related constructions are used in Chapter 2 for algebraic convenience. In our case we see that (3.6) corrects the inhomogeneous rescaling of $\mathcal{T}^i{}_{jk}$, endowing the galileons with a weight of -1. Thus, all relations in (3.3a) are reconciled with those in (3.3b). Finally, we tacitly convert matter fermions into bosons so as to preserve the SET

$$\begin{aligned}
T_{\mu\nu} &\equiv \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int \mathrm{d}^4x\sqrt{-g}L_{\mathrm{M}}(\Phi|g), \quad T_{\mu\nu} \stackrel{\text{an}}{=} h\tau_{(\mu\nu)}, \\
\tau^\mu{}_\nu &\equiv h_k{}^\mu\frac{\delta}{\delta h_k{}^\nu}\int \mathrm{d}^4x bL_{\mathrm{M}}(\Phi,\Psi,h,A) \equiv -b^k{}_\nu\frac{\delta}{\delta b^k{}_\mu}\int \mathrm{d}^4x bL_{\mathrm{M}}(\Phi,\Psi,h,A), \\
\sigma^\mu{}_{ij} &\equiv -\frac{\delta bL_{\mathrm{M}}(\Phi,\Psi,h,A)}{\delta A^{ij}{}_\mu} = 0,
\end{aligned} \tag{3.7}$$

note that the spin tensor density $\sigma^\mu{}_{ij}$ is *neglected* in this approximation.

At this point we are ready to derive the specific G_2 and G_4 which facilitate (3.2). Throughout the $\mathrm{PGT}^{\mathrm{q},+}$ equations, the nine Maxwell-like couplings appear exclusively in *five* linear combinations under the SCP, as given in Eq. (B.24a). These *physical* couplings are insensitive to e.g. a Gauss–Bonnet variation $4\delta\alpha_1 = -\delta\alpha_3 = 4\delta\alpha_6$, which is topological in $d \leq 4$. An application of the minisuperspace method is sufficient to obtain the required mapping from (3.2) to (3.1). Following (2.35) in a slightly different notation (i.e. taking u for S) we take $\mathrm{d}s^2 = u^2(\mathrm{d}t^2 - v^2\mathrm{d}\mathbf{x}^2)$, where the flat FLRW interval in (3.4) is recovered by taking $u \mapsto 1$ and $v \mapsto a$. The analogue defined in (3.6) corresponds to the following choices of gauge from (2.36) and Eqs. (2.40a) to (2.40c), in a further abuse of notation which assumes the holonomic and anholonomic bases to be aligned

$$b^i{}_\mu \stackrel{\text{an}}{=} u\Big(v\big(\delta^i_\mu - (\hat{\boldsymbol{e}}_t)^i(\hat{\boldsymbol{e}}_t)_\mu\big) + (\hat{\boldsymbol{e}}_t)^i(\hat{\boldsymbol{e}}_t)_\mu\Big), \quad A^{ij}{}_\mu \stackrel{\text{an}}{=} uv(\hat{\boldsymbol{e}}_t)^k\Big(\phi\delta^{[j}_\mu\delta^{i]}_k - \frac{1}{2}\psi\epsilon_{\mu k}{}^{ij}\Big). \tag{3.8}$$

The gauge fields in (3.8) are then substituted into (2.16) and (2.17), and then into (3.2). The Maxwell-like couplings defined in (B.24a), along with a minimal addition of surface terms (including the Gauss–Bonnet

[4] To avoid a notational clash with the Riemannian Ricci scalar in this chapter, we use $a \equiv R/R_0$.

derivative) then reduce this to

$$
\begin{aligned}
L_{\mathrm{T}} \overset{\text{an}}{=} & \Big(\frac{1}{2}{m_{\mathrm{p}}}^2\upsilon_2 + \sigma_3\phi^2 + \frac{1}{2}(\sigma_3-\sigma_2)\psi^2\Big)\big[6v^3(\partial_t u)^2 + 12uv^2\partial_t u\partial_t v + 6u^2v(\partial_t v)^2\big] \\
& + 12\sigma_3\Big[uv^3\phi\partial_t u + \frac{1}{2}u^2v^3\partial_t\phi + u^2v^2\phi\partial_t v\Big]\partial_t\phi + 6(\sigma_3-\sigma_2)\Big[uv^3\psi\partial_t u + \frac{1}{2}u^2v^3\partial_t\psi + u^2v^2\psi\partial_t v\Big]\partial_t\psi \\
& + 4\sigma_1\big(\psi^2-\phi^2\big)\Big[\frac{3}{2}u^2v^3\phi\partial_t u + \frac{3}{2}u^3v^2\phi\partial_t v + \frac{3}{2}u^3v^3\partial_t\phi\Big] + 3{m_{\mathrm{p}}}^2(\alpha_0+\upsilon_2)\big[u^2v^3\phi\partial_t u + u^3v^2\phi\partial_t v\big] \\
& + \frac{3}{4}u^4v^3\big[2\sigma_3\phi^4 - 4\sigma_2\phi^2\psi^2 + 2\sigma_3\psi^4 + {m_{\mathrm{p}}}^2(\alpha_0+\upsilon_2)\phi^2 - {m_{\mathrm{p}}}^2(\alpha_0-4\upsilon_1)\psi^2\big] \\
& + L_{\mathrm{M}}(\Phi,\Psi|u,v,\phi,\psi). \qquad (3.9)
\end{aligned}
$$

A naïve ansatz restricts to *polynomial* G-functions, but inspection of (3.9) reveals that this is only viable up to surface terms if $\alpha_0+\upsilon_2=\sigma_1=0$. These constraints eliminate terms of first-order in $\partial_t\phi$ and H from the penultimate line of (3.9), and so from the E.o.Ms. Such terms are non-canonical, but can be included (and the constraints removed) by extending (3.1) to $L_{\mathrm{T}}\mapsto L_{\mathrm{T}}+\Delta L_{\mathrm{T}}$, where

$$
\Delta L_{\mathrm{T}} = \big[{G_6}^{\phi}(\phi,\psi)\nabla_\mu\phi + {G_6}^{\psi}(\phi,\psi)\nabla_\mu\psi\big]B^\mu + m_{\mathrm{p}}\big({m_{\mathrm{p}}}^2 - B_\mu B^\mu\big)\chi. \qquad (3.10)
$$

The neutral vector B^μ and scalar χ may be thought of as gravitational *spurions*: they constrain the theory by singling out a preferred timelike vector under the SCP without breaking general covariance in the action [198]. The spurions are generally non-dynamical and are integrated out directly such that (3.10) merely renormalises G_2. Writing out the final G-functions explicitly, the full MA of (3.2) is

$$
\begin{aligned}
L_{\mathrm{T}} = & \Big[\frac{1}{2}{m_{\mathrm{p}}}^2\upsilon_2 + \sigma_3\phi^2 + \frac{1}{2}(\sigma_3-\sigma_2)\psi^2\Big]R + 12\Big[\sigma_3X^{\phi\phi} + \frac{1}{2}(\sigma_3-\sigma_2)X^{\psi\psi}\Big] + \sqrt{|J_\mu J^\mu|} \\
& + \frac{3}{4}{m_{\mathrm{p}}}^2\big[(\alpha_0+\upsilon_2)\phi^2 - (\alpha_0-4\upsilon_1)\psi^2\big] + \frac{3}{2}\big(\sigma_3\phi^4 - 2\sigma_2\phi^2\psi^2 + \sigma_3\psi^4\big) + L_{\mathrm{M}}(\Phi|g), \qquad (3.11\text{a}) \\
J_\mu \equiv & \, 4\sigma_1\psi^3\nabla_\mu(\phi/\psi) - {m_{\mathrm{p}}}^2(\alpha_0+\upsilon_2)\nabla_\mu\phi. \qquad (3.11\text{b})
\end{aligned}
$$

Further surface terms distinguish the minisuperspace Lagrangian of (3.11b) from (3.9), and a straightforward calculation confirms that the E.o.Ms coincide with those of $\mathrm{PGT}^{\mathrm{q},+}$ under the spatially-flat SCP.

3.2.4 First impressions

Noting in what follows that $\sqrt{|J_\mu J^\mu|}$ carries an implicit factor of $\mathrm{sgn}(J^0)$ for continuity [199], a straightforward calculation confirms that (3.11a) and (3.2) are dynamically coincident under the spatially-flat SCP. In this chapter we will not consider inhomogeneous applications, e.g. to acoustic stability. Various features of the MA are already apparent at the Lagrangian level. Since G_4 is not constant, ϕ and ψ are non-minimally coupled to R, thus the MA has been unwittingly but naturally constructed in the *Jordan* conformal frame (JF). It will prove convenient later to transform to the *Einstein* frame (EF), but since the EF derives its meaning from the artificial context of the MA, we cannot take it to be physical. Equivalently, to work at the usual level of the $\mathrm{PGT}^{\mathrm{q},+}$ equations is to work in the JF of the MA *and know no better*. While counter-intuitive, we find this picture to be unavoidable [200]. A scale-invariant $\mathrm{PGT}^{\mathrm{q},+}$ sets $\alpha_0=\upsilon_1=\upsilon_2=0$, reducing the MA to a *manifestly* conformal field theory [192]. In our minimal formulation, this would restrict to a pure radiation cosmology (see e.g. [140]), but we note that various Higgs-like scale symmetry-breaking extensions to the gauge theory have been proposed [163, 201, 202].

3.3 Applications

3.3.1 Application to established theories

Before addressing the novel theories, we will analyse some 'conventional' $\mathrm{PGT}^{\mathrm{q},+}$s with non-dynamical $A^{ij}{}_{\mu}$. Consider the representative two-parameter theory

$$L_{\mathrm{T}} = -\frac{1}{2}{m_{\mathrm{p}}}^2\alpha_0\mathcal{R} + \frac{1}{2}{m_{\mathrm{p}}}^2\beta\mathbb{T} + L_{\mathrm{M}}(\Phi,\Psi|h,A), \tag{3.12}$$

i.e. a linear combination of $\mathcal{R}$ and the *teleparallel* term $\mathbb{T} \equiv \frac{1}{4}\mathcal{T}_{ijk}\,\mathcal{T}^{ijk} + \frac{1}{2}\mathcal{T}_{ijk}\,\mathcal{T}^{jik} - \mathcal{T}_i\,\mathcal{T}^i$, with the MA

$$L_{\mathrm{T}} = -\frac{1}{2}{m_{\mathrm{p}}}^2\beta R + {m_{\mathrm{p}}}^2(\beta-\alpha_0)\Big[\sqrt{2|X^{\phi\phi}|} - \frac{3}{4}\phi^2 + \frac{3}{4}\psi^2\Big] + L_{\mathrm{M}}(\Phi|g). \tag{3.13}$$

We see that the MA is a linear combination of R, a quadratic *Cuscuton* ϕ with E.o.M $\phi = -2H$, and a non-dynamical mass which sets $\psi = 0$. By (3.6) we will have $U = Q = 0$. As a general principle, the *Cuscuton* is a non-dynamical constraint field, and preserves the form of the usual Friedmann equations of GR that follow from R. This can be seen by substituting ϕ into the $g_{\mu\nu}$ equation of (3.13) [184]. ECT theory is equivalent to GR when the spin tensor vanishes, and is defined by $\alpha_0 = 1$ and $\beta = 0$ in (3.12) [203]. Remarkably, this eliminates R from (3.13) entirely, so that $\mathcal{R}$ is represented purely by the *Cuscuton*. If $\beta \neq 0$, the admixture of $\mathbb{T}$ in (3.12) leads to R–*Cuscuton* contributions in (3.13) which *exactly cancel* in the $g_{\mu\nu}$ equation. However, true teleparallelism, with $\beta = 1$ and $\alpha_0 = 0$ is also dynamically equivalent to GR if PGT curvature (as defined in (2.16)) *vanishes* identically [204, 205, 67]. The constraint $\mathcal{R}^{ij}{}_{kl} \equiv 0$ is properly imposed via Lagrange multiplier fields [67], but in practice this just restricts $A^{ij}{}_{\mu}$ to a pure gauge (the *Weitzenböck connection*) and fixes $\phi \equiv \psi \equiv 0$. By (3.6) we will then have $Q \equiv 0$ and $U \equiv 3H$. Since the *Cuscuton* is now eliminated, $\mathbb{T}$ is represented purely by R, and the expected equivalence to GR is immediate. We will return to the use of teleparallel multipliers in Chapter 5.

3.3.2 Application to novel theories

We now focus in on extending our results from Chapter 2. Following on from our discussion in Section 2.3, for momentum k^i the graviton and roton propagators of a generic $\mathrm{PGT}^{\mathrm{q},+}$ may approach the UV as k^{2N_h} and k^{2N_A}, where $N_h, N_A \leq 0$ are some integers, and the even powers are expected of bosons. In such a theory, a diagram may have E_h external graviton and E_A external roton lines. Also, there will be V_{nm} vertices with n graviton and m roton valences, and whose coupling constant has some (low) mass dimension C_{nm} supplied by the appearance of m_{p} in (3.2). By considering the perturbative structure of (3.2) and applying the usual topological identity that relates the number propagators, vertices and loops [207], one eventually arrives at the following formula for the superficial divergence D of the diagram

$$D = 4 - (2+N_h)E_h - (2+N_A)E_A - \sum_{n,m}\left[C_{nm} - 2n(2+N_h) - 2m(1+N_A)\right]V_{nm}. \tag{3.14}$$

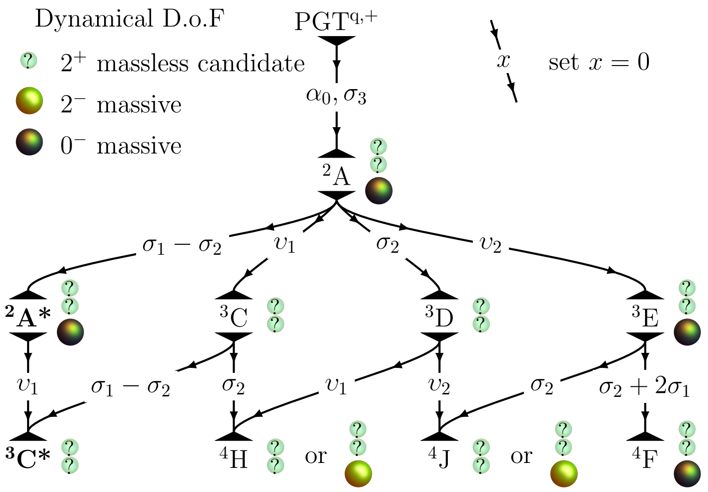


Fig. 3.1 Cosmologies and associated particle content of the novel theories, obtained by 'zooming in' to the viable part of Fig. 2.1 (see [153] and literature comparisons in [152] and Chapter 2). In the weak, free-field limit, certain cases of $\mathrm{PGT}^{\mathrm{q},+}$ are unitary and PCR. These cases contain propagating irreps of $\mathrm{SO}^{+}(1,3)$, i.e. D.o.F of spin-parity J^P. For massless D.o.F, the propagator poles associated with any contributing J^P sectors are degenerate at the origin of k-space. Since this leads to ambiguity, we restrict to cases which *do not preclude* the two 2^+ polarisations of the graviton (which should be unique [206]). The cases are grouped into cosmological *classes*, of which we consider Class ^{2}A* and Class ^{3}C*.

The strong PCR criteria $N_h = -2$ and $N_A = -1$ are then *suggestive* of perturbative renormalisability. If these criteria are met, one can see from (3.14) that any diagram appearing at high enough loop order or with sufficiently many external lines should superficially converge, unlike in Fig. 1. While such a diagram may still be divergent in practice, there is some hope that this divergence may result from the incorporation of a finite number of primitively divergent diagrams. The novel cases in [152, 153] are defined by linear constraints on the ten $\mathrm{PGT}^{\mathrm{q},+}$ parameters. These constraints structurally alter the saturated propagator, obtained by inverting the linearised, matter-free Lagrangian in (3.2), so as to *effectively* satisfy these criteria. As noted in Section 2.3, the criteria may be safely relaxed for modes which become non-propagating in the UV; for a full discussion of this matter the reader is referred to [153, 158].

The SCP groups the cases into *classes*, some of which are shown in Fig. 3.1. The constraint $\alpha_0 = 0$ marks a complete break with ECT theory: one is left only with quadratic invariants which have no EFT interpretation as loop corrections to the PGT Ricci scalar $\mathcal{R}$. The further constraint $\sigma_3 = 0$ then triggers the k-screening mechanism, in which the physical spatial curvature $k \in \{\pm 1, 0\}$ is *eliminated* from the $\mathrm{PGT}^{\mathrm{q},+}$ equations: as shown in Chapter 2, a hyperspherical, hyperbolic or simply flat choice of Universe does not affect the background dynamics. The description of such classes as offered by the MA is thus *not limited* by our earlier assumption of spatial flatness in (3.4).

We consider Class ^{2}A*, defined by the further constraint $\sigma_2 = \sigma_1$ (recall that Class ^{3}C* will always be the special case $\upsilon_1 = 0$). We next set $\sigma_1 < 0$ (no ghost) and $\upsilon_1 < 0$ (no tachyon): these unitarity conditions are translated from [153]. They may also be read off from (3.11a) near the vacuum $R = \phi = \psi = 0$, once the defining constraints are imposed. We finally take a third condition $\upsilon_2 < 0$ by analogy to the Einstein-Hilbert Lagrangian, although this is not listed in [153]. A conformal transformation Ω takes the MA of Class ^{2}A* into the EF. Following the conventions of e.g. Brans–Dicke theory [208], we then partly recanonicalise the MA through two new fields $\zeta(\phi,\psi)$ and $\xi(\psi)$

$$L_{\rm T} = -\frac{1}{2}{m_{\rm p}}^2 R + X^{\xi\xi} + {m_{\rm p}}^2\omega(\xi)^3\sqrt{|X^{\zeta\zeta}|} - V(\xi) + \frac{3}{4}{m_{\rm p}}^2\omega(\xi)^4\zeta^2 + L_{\rm M}(\Phi,\xi|g), \tag{3.15a}$$

$$V(\xi) \equiv -\frac{4\upsilon_1}{3\sigma_1\upsilon_2}{m_{\rm p}}^4\Big(1+\frac{1}{8}\omega(\xi)^2\Big)\Big(1+\frac{1}{2}\omega(\xi)^2\Big), \tag{3.15b}$$

$$\omega(\xi) \equiv \sqrt{\left|3\cosh\left(\sqrt{2/3}\;\xi/m_{\rm p}\right)-5\right|}. \tag{3.15c}$$

While (3.15) is strictly valid for the range

$$1 \leq 4\sigma_1 Q^2/\upsilon_2 {m_{\rm p}}^2 < 4, \tag{3.16}$$

we will use it to obtain physical results which are completely general, as may be confirmed directly from (3.11a). In fact, we will later see that the Universe is expected to lie in this range for most of its history anyway. Noting that $\Omega^2 = -\frac{4}{3\upsilon_2}\big(1+\frac{1}{8}\omega^2\big)$, it seems natural in what follows to take $\upsilon_2 = -4/3$: this choice was tacitly assumed in Section 2.5.5, and here it will be justified in stages. The 'conformal shift' ω now measures the degree to which the physical JF has strayed from the EF, and so mediates any ξ–Φ coupling. Note that ω also weights the field ζ, which is a quadratic *Cuscuton*. The field ξ is canonical, and in moving from Class ^{3}C* to Class ^{2}A* it acquires a potential V. Note that V traces back to the mass of ψ, which in turn corresponds to the massive 0^- D.o.F in Fig. 3.1. By inspection, V must act as a (quintessence) dark energy source, since $\upsilon_1/\sigma_1 > 0$. In the final sections we will make the nature of this dark energy more concrete, using the ζ E.o.M as a heuristic

$$\omega^2\big(\sqrt{2}\partial_\xi\omega\partial_t\xi + \sqrt{2}\omega H - \omega^2\zeta\big) = 0. \tag{3.17}$$

3.3.3 Negative screened dark energy

By analogy to (3.13), suppose that the *Cuscuton* obeys $\zeta \propto H$, which was its 'minimally-coupled' behaviour. This is possible if the last two terms in (3.17) cancel, whereupon the decay of ξ *stalls* above the natural vacuum of V at constant conformal shift $\omega = \sqrt{2}H/\zeta$. This solution has the following utility if the physical JF matter Lagrangian contains only a bare cosmological constant $L_{\rm M}(\Phi|g) = -{m_{\rm p}}^2\Lambda_{\rm b}$. Accelerated expansion is difficult to drive with $\Lambda_{\rm b} < 0$ in many gravitational theories. This can make them hard to reconcile with attractive, more fundamental theories [209–212]. If $L_{\rm M}(\Phi,\xi|g) = -{m_{\rm p}}^2\Lambda_{\rm b}\big(1+\frac{1}{8}\omega^2\big)^2$ and $\omega = \sqrt{2}H/\zeta$ are substituted into the remaining E.o.Ms of (3.15), one can straightforwardly solve for ξ and H in the EF. In the physical JF this gives $Q^2 = 2\Lambda_{\rm b}/3\upsilon_1$, and $H^2 = \Lambda/3$, where the effective cosmological constant is $\Lambda = \upsilon_1 {m_{\rm p}}^2/2\sigma_1$. Remarkably therefore, a *negative* $\Lambda_{\rm b}$ is required, yet *screened* from the de Sitter expansion rate.

To verify the stability of the de Sitter solution, we employ another product of the MA: the powerful dynamical systems theory of scalar-tensor inflation [199, 213]. We view ξ as a canonical inflaton, whose 'total potential' is $V_{\rm T} \equiv V + L_{\rm M}(\Phi,\xi|g)$. It is possible to encode all E.o.Ms as an autonomous, first-order system in the dimensionless variables

$$x^2 \equiv \frac{{m_{\rm p}}^2(\partial_t\xi)^2}{6H^2}, \quad y^2 \equiv \frac{V_{\rm T}}{3{m_{\rm p}}^2H^2}, \tag{3.18}$$

which are the comoving Hamiltonian coordinates of ξ. In order to obtain this form, we further define intermediate dimensionless variables

$$z^2 \equiv \frac{m_{\text{p}}{}^2\omega^4\zeta^2}{4H^2}, \quad \lambda \equiv -\frac{m_{\text{p}}\partial_\xi V_{\text{T}}}{V_{\text{T}}}, \quad \mu \equiv \omega. \tag{3.19}$$

Note that x, y and λ are conventional parameters in the literature, while μ is defined for convenience and z is somewhat analogous to the conventional matter parameter [199, 213]. From (3.15), the ξ equation (or alternatively the pressure–$g_{\mu\nu}$ equation) combined with the derivative of the ζ equation (3.17) can be expressed as a coupled first-order system in terms of these variables

$$\begin{aligned}
\partial_\tau x = -\Big[&x\big(2\sqrt{3}\lambda\mu^3 xy^2 + 4z^2\big((\mu^4-8)z^2 - 2(\mu^4-4)y^2 - 8\big) + \sqrt{2}\mu\big(\sqrt{3}\lambda\mu^3 xy^2z^2 \\
&+ \mu^4(y-z)^2(y+z)^2 - 16(y-z)(y+z)\big(y^2-z^2-1\big) \\
&+ 2\mu^2\big(y^2-1\big)\big(3y^2-z^2\big)\big)\big)\Big]\Big/\Big[\mu^3\big(\sqrt{2}\big((2+\mu^2)y^2 - \mu^2z^2 - 2\big) - 4\mu z^2\big)\Big],
\end{aligned} \tag{3.20a}$$

$$\begin{aligned}
\partial_\tau y = y\Big[&\mu^2\big(\sqrt{3}\lambda x\big(2-2y^2+\mu^2z^2\big) - 4\mu z^2\big(3-3y^2+z^2\big)\big) - \sqrt{2}\big(2\mu^2\big(y^2-1\big)\big(3y^2-z^2-3\big) \\
&+ \sqrt{3}\lambda\mu^3 x\big(y^2-2\big)z^2 - 16\big(1-y^2+z^2\big)^2 + \mu^4\big(y^4+z^2\big(3+z^2\big) \\
&- y^2\big(1+2z^2\big)\big)\big)\Big]\Big/\Big[\mu^2\big(\sqrt{2}\big((2+\mu^2)y^2-\mu^2z^2-2\big) - 4\mu z^2\big)\Big],
\end{aligned} \tag{3.20b}$$

where the dimensionless (Hubble-normalised) time is $\mathrm{d}\tau \equiv H\mathrm{d}t$. In order to obtain the autonomous system in x and y we must eliminate λ, μ and z from (3.20a) and (3.20b). Using (3.15b) and (3.15c), it is possible to solve for λ in terms of μ

$$\lambda = -\frac{\Big[4\Big(2\Lambda_{\text{b}}+5\frac{\upsilon_1}{\sigma_1}m_{\text{p}}{}^2\Big)+\Big(\Lambda_{\text{b}}+4\frac{\upsilon_1}{\sigma_1}m_{\text{p}}{}^2\Big)\mu^2\Big]\sqrt{\big(2+\mu^2\big)}}{\Big[8\Big(\Lambda_{\text{b}}+\frac{\upsilon_1}{\sigma_1}m_{\text{p}}{}^2\Big)+\Big(\Lambda_{\text{b}}+4\frac{\upsilon_1}{\sigma_1}m_{\text{p}}{}^2\Big)\mu^2\Big]\sqrt{3\big(1+\frac{1}{8}\mu^2\big)}}. \tag{3.21}$$

Note that (3.21) explicitly incorporates both the bare cosmological constant Λ_{b} and our central combination $\upsilon_1 m_{\text{p}}{}^2/\sigma_1$. As emphasised above, these quantities should be considered on an equal footing. Next, the ζ equation reduces to a quartic in μ

$$\big(x^2-1\big)\mu^4 + 2\sqrt{2}z\mu^3 + 2\big(5x^2-z^2\big)\mu^2 + 16x^2 = 0. \tag{3.22}$$

Finally, z is solved for x and y by the density–$g_{\mu\nu}$ equation

$$x^2 + y^2 - z^2 = 0, \tag{3.23}$$

revealing that the physical portions of the phase space are *expelled* from the unit disc. If z were a 'conventional' matter parameter (i.e. proportional to a density which is obedient to the weak energy condition), the phase space would be *confined* to the unit disc. This more holistic picture, in which all critical points $\partial_\tau x = \partial_\tau y = 0$ are visible, may be reached by taking a simple Möbius transform of the phase space. The quartic roots of (3.22) cause the fully autonomous system to be highly unwieldy. This is a natural consequence of explicitly encoding the *Cuscuton* constraint in the Class ^{2}A* and Class ^{3}C* theories, rather than a generic limitation of the MA in (3.11a). Returning at last to the question of stability, the de Sitter solution outlined above is then found to be a stable critical point in this system, as illustrated in Fig. 3.2.

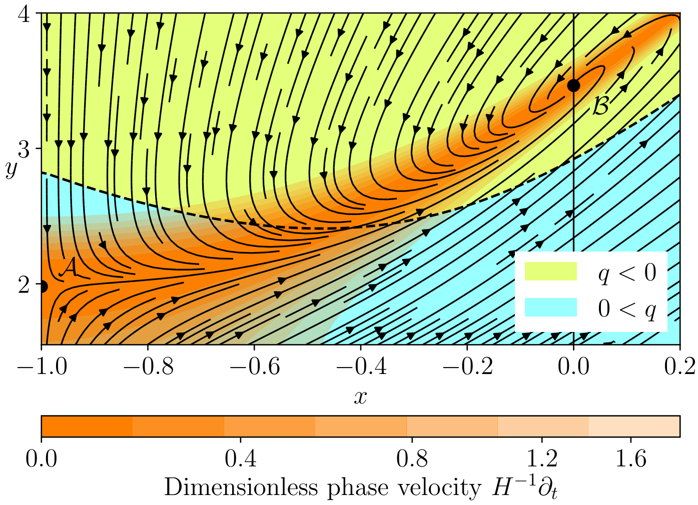


Fig. 3.2 Partial phase portrait of Class ^{2}A*, with *negative* bare cosmological constant $\Lambda_\text{b} = -0.48{m_\text{p}}^2$. The saddle $\mathcal{A}$ deflects the Universe towards the de Sitter attractor $\mathcal{B}$ in the inflationary region where it feels a *positive* effective $\Lambda = 0.1{m_\text{p}}^2$, all in the physical JF. The EF deceleration parameter is $1 + q = -\partial_t H/H^2$. Hamiltonian coordinates y and x describe the 0^- torsional mode. Phase velocity reflects elapsing Hubble-times.

While $\zeta \propto H$ may describe our late Universe if $\Lambda_\text{b} < 0$, it is not clear that it will be self-consistent in a matter-dominated epoch. Therefore, we will next consider a family of solutions which naturally describe the whole expansion history.

3.3.4 Generally viable dark energy

The 'generally viable' solution to (3.17) occurs at vanishing conformal shift $\omega = 0$, where the EF and physical JF *coincide*. We previously termed this the CS in Chapter 2. The CS of Class ^{3}C* reduces (3.15a) to GR by inspection; Class ^{2}A* differs from this through the constant stalled potential V. The stalled ξ fixes ${Q_\text{cor}}^2 \equiv -{m_\text{p}}^2/3\sigma_1$. If the Universe is reasonably assumed to follow the CS closely, then Q should not stray too far from this critical value, which is fortunately the lower bound in (3.16). The equivalence of conformal frames is guaranteed by our earlier condition $\upsilon_2 = -4/3$. Broadly speaking, this has the same effect as fixing Einstein's κ in GR.

The stability of the CS should be verified for all matter in LCDM including the conventional $\Lambda_\text{b} \geq 0$, but the earlier dynamical systems approach is impractical in this case. Such matter may be characterised by E.o.S $\rho = wP$, diluting away as $\rho \propto a^{-3(1+w)}$. For any dominant matter, a straightforward perturbation around the CS is equivalent to adding an effective fluid $\rho \mapsto \rho + \rho_\text{eff}$ to GR – note that we set $\rho_\text{eff} = \varepsilon\delta\rho + \mathcal{O}(\varepsilon^2)$ to connect with the perturbative formalism in Section 2.5.5. The effective E.o.S parameter *tracks* the dominant w according to

$$w_\text{eff}(w) \equiv \frac{1}{2}(w+1) - \frac{1}{6}\sqrt{9w^2+3}, \quad -1 \leq w \leq \frac{1}{3}, \tag{3.24}$$

The ρ_eff becomes increasingly sub-dominant (and the CS is stable) when $w_\text{eff}(w) > w$; the only exception is co-dominant dark radiation, since $w_\text{r,eff} \equiv w_\text{eff}(1/3) = 1/3$. The possible utility of this dark radiation in shrinking the sound horizon at recombination (and raising the early-Universe inference of h) is discussed in Chapter 2, while (3.24) nicely fills in the gaps of our previous model as given in (2.9). Note that the *effective* fluid need not satisfy the weak energy condition by itself. This strengthens the justification of (3.16), since a value of Q below the lower bound would manifest as $\rho_{\text{eff}} < 0$, i.e. a negative dark radiation fraction which would exacerbate the Hubble tension. Finally, the stalled V readily gives an effective $\Lambda = \Lambda_{\text{b}} + \upsilon_1 m_{\text{p}}{}^2/\sigma_1$.

3.4 Closing remarks

We constructed in Eqs. (3.11a) and (3.11b) a non-canonical bi-scalar-tensor theory, the *metrical analogue* (MA) which lays bare the rich IR background cosmology of $\text{PGT}^{\text{q},+}$. It is natural that the theory explicitly includes only the cosmological 0^+ and 0^- torsion sectors, rather than all 20 D.o.Fs native to PGT. As a consequence, portions of both the IR and UV are necessarily lost. In particular, it is evident that no parameter constraint may be applied to the MA itself to render it perturbatively renormalisable. This follows since the MA is an explicit extension of GR by scalar D.o.Fs, and lacks any of the expected quadratic curvature invariants. However, we see no reason why this should affect the anticipated renormalisability of the underlying $\text{PGT}^{\text{q},+}$. Rather, it is interesting to consider how the quadratic and linear invariants of $\text{PGT}^{\text{q},+}$ are allocated to the linear invariant of the MA. Tellingly, it is teleparallelism and the other quadratic theories which inherit the Einstein–Hilbert Lagrangian, while ECT theory is relegated to a *Cuscuton*. We verified that the Friedmann equations are recovered in both cases. This illustrates, in the context of our introductory discussion, the naturalness of *quadratic* PGT Lagrangia.

Our analysis in this chapter of the MA phenomenology was not intended to be exhaustive. Particularly, our approach invites inflationary applications in the early Universe, and extension to Weyssenhoff fluids through a non-minimal ψ-coupling to modified matter sources [214]. A principle observation is that $\text{PGT}^{\text{q},+}$, when expressed in scalar-tensor form, contains a non-canonical term which may often be interpreted as a *Cuscuton* field. While this interpretation offers theoretical support to the *Cuscuton*, we note that it is not unique. For instance, it is evident from (3.10) that by alternatively integrating out a galileon the MA would contain a neutral vector. Specifically, the physics is basically equivalent (as is the *Cuscuton* itself) to the cosmological model of *Lorentz-violating vector fields* [215].

In this chapter we focused on late-Universe dark energy in the superficially healthy cases of $\text{PGT}^{\text{q},+}$ proposed in Chapter 2. The proposed emergent $\Lambda = \Lambda_{\text{b}} + \upsilon_1 m_{\text{p}}{}^2/\sigma_1$ still does not address the 'strong' cosmological constant problem [30, 186]. Let us assume a 'non-gravitating vacuum' $\Lambda_{\text{b}} = 0$ [216, 30, 217]. CMB-inference fixes $\Lambda = (7.15 \pm 0.19) \times 10^{-121}\,\text{m}_{\text{p}}{}^2$ [24], with some (slight) shift expected from any dark radiation we may choose to add [117, 119]. The requisite $\upsilon_1/\sigma_1 \sim 1 \times 10^{-121}$ then reveals an apparent hierarchy. We tentatively observe that the hierarchy appears less severe in the scale-invariant eWGT counterpart, since the $\sim 4.1\,\text{Gpc}$ Hubble horizon endows specific physical $\text{eWGT}^{\text{q},+}$ couplings with a natural length scale [92]. This builds the case for a future extension of the systematic analysis in [152, 153, 158] to $\text{eWGT}^{\text{q},+}$, whose propagator is currently unexplored.

In a conservative summary, the Class $^2\text{A}^*$ theory not only *matches* the GR background, but can provide dark radiation and (hierarchical) dark energy. Unlike GR [49], the perturbative renormalisability of this unitary theory is not precluded by a simple power counting [152, 153]. The 0^- torsional mode must survive averaging over homogeneous comoving scales of $\gtrsim 300\,\mathsf{h}^{-1}\,\text{Mpc}$ [19, 20]. This mode has yet to be constrained, even in an Earth-based laboratory [124, 218, 177], and its strength is not separable here from the σ_1 or υ_1 couplings. Indeed, the expansion history only determines υ_2 and υ_1/σ_1, which translate to the two freedoms in Lovelock's theorem.

We will return to the phenomenology of this theory in Chapter 5. In the next chapter we turn to some of the theoretical challenges faced by the new $\mathrm{PGT}^{\mathrm{q},+}$s in general.

Chapter 4

Nonlinear Hamiltonian analysis of the new gauge theories

Abridged from W. E. V. Barker, A. N. Lasenby, M. P. Hobson and W. J. Handley, Physical Review D *in press.*, arXiv:2101.02645 [gr-qc].
Accepted content also appears in Appendices C.1 to C.4.

4.1 Introduction

The unitarity and PCR requirements placed on the $\mathrm{PGT}^{\mathrm{q},+}$ in [152, 153] restrict the dynamical structure only at *linear* order. Since they were found in Chapters 2 and 3 to sometimes coincide serendipitously with the *nonlinear* phenomenology, it seems appropriate next to consider how these requirements themselves generalise at nonlinear order. In this chapter therefore we will probe the *nonlinear Hamiltonian* structure of the new theories. As a higher-spin gauge theory, the $\mathrm{PGT}^{\mathrm{q}+}$ (3.2) is always singular: this degeneracy of the kinetic Hessian greatly complicates the Lagrangian analysis, incentivising the Hamiltonian approach. By implementing the algorithm of Dirac and Bergmann, we are guaranteed to obtain all propagating D.o.F, along with all constraints [219]. In the linearised theory, this is especially easy, and allows us to verify the particle spectra and unitarity of the cases obtained in [152, 153]. In the nonlinear case, the algorithm allows us to flag potentially fatal pathologies which develop under significant departures from Minkowski spacetime – if this spacetime is taken to be a vacuum, then the nonlinear regime is equivalent to that of strong fields. In particular, we rely on the simple 'health indicator' of modified gravity set out by Chen, Nester and Yo: *the number and type of constraints should not change in passing from the linear to nonlinear regimes* [220, 169]. The motivation for this criterion is twofold. Generically, a decrease in the number of constraints involves the activation of potentially *ghostly* fields [169]. Moreover, it may be that the nonlinear constraint structure is itself field dependent: this is thought to be associated with the propagation of *acausal* D.o.F [220]. Neither of these qualities is necessarily fatal unless shown to incur a physical ghostly or acausal D.o.F, but for the purposes of this particular study we will take the avoidance of them as being desirable.

In this chapter we will test Case 3, Case 17, Case 20, Case 24, Case ^{*5}25, Case ^{*6}26, Case 28 and Case 32, using the numbering of [153], with the numbering of cases previously discovered in [152] indicated by (*). These eight cases are most conducive to the Hamiltonian analysis. Specifically, these are the only cases whose primary constraints are not functions of the curvature. To our knowledge, this practical restriction does no more than to ease the evaluation of commutators. We therefore tentatively view the eight cases to be a *representative sample* of the 58 novel theories[1].

All eight cases fail the prescribed strong-field tests. In some sense, they do so more dramatically than those 'minimal' cases of $\mathrm{PGT}^{\mathrm{q}+}$ which were previously tested, due to the vanishing of mass parameters [169]. Based on these results, we find no evidence that the simultaneous imposition of the weak-field PCR and unitarity criteria remedy the questionable health of $\mathrm{PGT}^{\mathrm{q}+}$ in the strong-field regime, as observed in [220, 168, 169]. If these findings turn out to be general, it would seem more efficient to perform future surveys of $\mathrm{PGT}^{\mathrm{q}+}$ in the strong-field regime *from the outset.*

Fortunately none of these eight cases were considered in the previous chapters. Indeed, by applying our previous methods we are also able to disqualify them on independent phenomenological grounds. Out of the eight cases, only Case 3 and Case 17 propagate massless modes consistent with long-range gravitational forces, yet their nonlinear cosmological equations are *non-dynamical.* We note in passing that, from an inspection of Table 2.1, these are the only two cases analysed in this chapter whose cosmologies were not already considered in Chapter 2. However, we do show that these cases are the degenerate limit of an otherwise viable and interesting class of torsion theories obtained by imposing two very simple constraints on the couplings of (3.2), whose background cosmology perfectly replicates that of Einstein's torsion-free gravity (1), conformally coupled to a scalar inflaton ξ

$$L_{\mathrm{T}} = -\frac{1}{2}{m_{\mathrm{p}}}^2 R + \frac{1}{12}\xi^2 R + X^{\xi\xi} - \frac{1}{2}{m_\xi}^2\xi^2 + L_{\mathrm{M}}. \tag{4.1}$$

Here, the inflaton has kinetic term $X^{\xi\xi} \equiv \frac{1}{2}\nabla_\mu\xi\nabla^\mu\xi$ and mass m_ξ. The cosmology resulting from (4.1) is not scale-invariant due to the mass term, which is fortunate for minimal coupling to cosmological matter. However, it is an interesting surprise that the non-minimal coupling should be exactly scale-invariant.

The phenomenological failure of Case 3 and Case 17 is of course *not* a necessary consequence of the linearised unitarity and PCR criteria, as demonstrated by the viable Case 2 and Case 16 in Chapters 2 and 3. Hamiltonian analysis of these viable theories is deferred to Chapter 5, since their primary constraints depend on curvature.

Despite our concerns about the strong-field regime, we are able to confirm the weak-field unitarity of all eight cases. We also obtain linearised dynamics which are consistent with the particle spectra found in [152, 153]. In addition, we are able to offer tighter bounds on the massless particle spectra, identifying the massless modes of Case 3 and Case 17 as *vector* excitations, rather than the expected *tensor* polarisations of the graviton.

The remainder of this chapter is set out as follows. In Section 4.2 we develop the Hamiltonian formulation of the ten-parameter theory (3.2). In Sections 4.3 and 4.4 we apply the Dirac–Bergmann algorithm to each of the linearised cases, and compare with the constraint structure of the nonlinear theories. In Section 4.5 we use efficient methods to show that even the cases with massless modes cannot support any Friedmann-like cosmology. Conclusions follow in Section 4.6.

[1] We mention that *none* of the eight cases are PCR in the most conservative sense of [156], i.e. all of them feature a J^P propagator whose momentum power is non-PCR in the IR, but which decouples in the UV.

Table 4.1 From the 58 unitary, PCR cases of (3.2), we consider the eight cases whose primary constraints do not depend on the Riemann–Cartan curvature, using the same conventions as in Table 2.1 but this time with the irreducible couplings of Eqs. (B.23j) to (B.23l). Note that the constraint $\hat{\alpha}_0 = 0$ is always implicit.

#	criticality equalities	unitary inequalities	0^-	0^+	1^-	1^+	2^-	2^+	D.o.F
Case 3	$\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_4 = \hat{\alpha}_6 = \hat{\beta}_2 = \hat{\beta}_1 + 2\hat{\beta}_3 = 0$	$\hat{\alpha}_3 < 0 \wedge \hat{\alpha}_5 < 0 \wedge \hat{\beta}_1 < 0$	●		○○		○	○○	● ○ ○
Case 17	$\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_3 = \hat{\alpha}_4 = \hat{\alpha}_6 = \hat{\beta}_2 = \hat{\beta}_1 + 2\hat{\beta}_3 = 0$	$\hat{\alpha}_5 < 0$	○		○○		○	○○	○ ○
Case 20	$\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_4 = \hat{\alpha}_5 = \hat{\alpha}_6 = 0$	$0 < \hat{\beta}_3 \wedge \hat{\alpha}_3 < 0$	●	○○	○○○	○○	○	○○	
Case 24	$\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_4 = \hat{\alpha}_6 = \hat{\beta}_1 = 0$	$0 < \hat{\beta}_3 \wedge \hat{\alpha}_3 < 0$	●	○○	○○○	○○			
Case *5 25	$\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_4 = \hat{\alpha}_5 = \hat{\alpha}_6 = \hat{\beta}_1 = 0$	$0 < \hat{\beta}_3 \wedge \hat{\alpha}_3 < 0$	●	○○	○○○	○○			●
Case *6 26	$\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_4 = \hat{\alpha}_5 = \hat{\alpha}_6 = \hat{\beta}_1 = \hat{\beta}_2 = 0$	$0 < \hat{\beta}_3 \wedge \hat{\alpha}_3 < 0$	●			○○			
Case 28	$\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_4 = \hat{\alpha}_6 = \hat{\beta}_1 = \hat{\beta}_2 = 0$	$0 < \hat{\beta}_3 \wedge \hat{\alpha}_3 < 0$	●		○	○○			
Case 32	$\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_4 = \hat{\alpha}_5 = \hat{\alpha}_6 = \hat{\beta}_2 = 0$	$0 < \hat{\beta}_3 \wedge \hat{\alpha}_3 < 0$	●		○○○	○○	○	○○	

4.2 Constrained Hamiltonian

4.2.1 Primary constraints and $3+1$

In order to transition to the constrained Hamiltonian picture [67, 219, 221], we first define the *canonical* momenta as follows

$$\pi_i{}^{\mu} \equiv \frac{\partial b L_{\mathrm{G}}}{\partial(\partial_0 b^i{}_{\mu})}, \quad \pi_{ij}{}^{\mu} \equiv \frac{\partial b L_{\mathrm{G}}}{\partial(\partial_0 A^{ij}{}_{\mu})}. \tag{4.2}$$

Following [152, 153], we will consider only the gravitational part of the Lagrangian, i.e. without any matter L_{M}. Since the field strengths (2.16) and (2.17) from which (3.2) is constructed make no reference to the velocities of $b^k{}_0$ and $A^{ij}{}_0$, the definitions (4.2) incur 10 primary constraints

$$\varphi_k{}^0 \equiv \pi_k{}^0 \approx 0, \quad \varphi_{ij}{}^0 \equiv \pi_{ij}{}^0 \approx 0, \tag{4.3}$$

so that the conjugate fields $b^k{}_0$ and $A^{ij}{}_0$ are non-physical. Notice that the *weak* equality is denoted by ($\approx$). The constraints (4.3) are a consequence of Poincaré gauge symmetry; their presence is independent of the couplings, and they are first class (FC). We refer to them as 'sure' primary, first class (sPFC) constraints. In order to systematically isolate the 'sure' non-physical fields, we introduce the $3+1$ (ADM) splitting of the spacetime, in which a spacelike foliation is characterised by timelike unit normal n_k. Any vector which refers to the local Lorentz basis may be split into components $\mathcal{V}^i = \mathcal{V}^{\perp} n^i + \mathcal{V}^{\bar{i}}$ which are respectively perpendicular and parallel to the foliation: parallel indices are always denoted with an overbar. In what follows, it is very useful to note the identities $b^{\bar{k}}{}_{\alpha} h_{\bar{l}}{}^{\alpha} = \delta^{\bar{k}}_{\bar{l}}$ and $b^{\bar{k}}{}_{\alpha} h_{\bar{k}}{}^{\beta} = \delta^{\beta}_{\alpha}$. The *lapse* function and *shift* vector are defined with reference to the non-physical part of the translational gauge field using this normal $N \equiv n_k b^k{}_0$, and $N^{\alpha} \equiv h_{\bar{k}}{}^{\alpha} b^{\bar{k}}{}_0$. The remaining momenta can be expressed in the 'parallel' form $\hat{\pi}_i{}^{\bar{k}} \equiv \pi_i{}^{\alpha} b^k{}_{\alpha}$ and $\hat{\pi}_{ij}{}^{\bar{k}} \equiv \pi_{ij}{}^{\alpha} b^k{}_{\alpha}$. In order to reveal the Hamiltonian structure of the theory as naturally as possible, the Lagrangian in (3.2) is best written in the irreducible form

$$L_{\mathrm{T}} = -\frac{1}{2}\hat{\alpha}_0 m_{\mathrm{p}}{}^2 \mathcal{R} + \sum_{I=1}^{6} \hat{\alpha}_I \mathcal{R}^{ij}{}_{kl}{}^{I}\hat{\mathcal{P}}_{ij}{}^{kl}{}_{nm}{}^{pq}\mathcal{R}^{nm}{}_{pq} + m_{\mathrm{p}}{}^2 \sum_{M=1}^{3} \hat{\beta}_M \mathcal{T}^i_{jk}{}^{M}\hat{\mathcal{P}}_i{}^{jk}{}_l{}^{nm}\mathcal{T}^l_{nm} + L_{\mathrm{M}}. \tag{4.4}$$

where the nine operators ${}^{I}\hat{\mathcal{P}}^{\cdots}{}_{\cdots}$ and ${}^{M}\hat{\mathcal{P}}^{\cdots}{}_{\cdots}$ project out all the irreducible representations of $\mathrm{SO}^{+}(1,3)$, in no particaular order, from the field strengths. For the details of these projections, including the linear translation between the quadratic couplings of (3.2) and (4.4), see Appendix C.1. Within the field strengths, the $3+1$ splitting is used again to separate out the fields $b^{k}{}_{0}$ and $A^{ij}{}_{0}$ (which are non-physical) and the velocities of all fields (which are non-canonical)

$$\mathcal{T}^{i}{}_{kl} = \mathcal{T}^{i}{}_{\overline{kl}} + 2n_{[k}\mathcal{T}^{i}{}_{\perp\overline{l}]}\ , \quad \mathcal{R}^{ij}{}_{kl} = \mathcal{R}^{ij}{}_{\overline{kl}} + 2n_{[k}\mathcal{R}^{ij}{}_{\perp\overline{l}]}, \tag{4.5}$$

where such variables are confined to the second term in each case. We are concerned with theories of the quadratic $L_{\mathrm{G}} \sim \mathcal{R}^2 + m_{\mathrm{p}}{}^2\mathcal{T}^2$ form (i.e. $\hat{\alpha}_0 = 0$), under the source-free condition $L_{\mathrm{M}} = 0$. Substituting (4.4) into (4.2) and using Eq. (4.5), we find that the parallel momenta can be neatly expressed as functions of the field strengths

$$\frac{\hat{\pi}_{i}{}^{\overline{k}}}{J} \equiv \frac{\partial L_{\mathrm{T}}}{\partial \mathcal{T}^{i}{}_{\perp\overline{k}}} = 4m_{\mathrm{p}}{}^{2}\sum_{M=1}^{3}\hat{\beta}_{M}\,{}^{M}\hat{\mathcal{P}}_{i}{}^{\perp\overline{k}}{}_{n}{}^{ml}\mathcal{T}^{n}{}_{ml}\ , \quad \frac{\hat{\pi}_{ij}{}^{\overline{k}}}{J} \equiv \frac{\partial L_{\mathrm{T}}}{\partial \mathcal{R}^{ij}{}_{\perp\overline{k}}} = 8\sum_{I=1}^{6}\hat{\alpha}_{I}\,{}^{I}\hat{\mathcal{P}}_{ij}{}^{\perp\overline{k}}{}_{mn}{}^{pq}\mathcal{R}^{mn}{}_{pq}, \tag{4.6}$$

where the measure $J \equiv b/N$ on the foliation is strictly physical, since $b^{k}{}_{0}$ is divided out by N.

Writing the parallel momenta in this form facilitates the identification of further primary constraints. From the first relation in (4.6), we find that the 12 translational parallel momenta decompose into four irreducible representations of SO(3). Using the spin-parity notation of [168, 169] we write these as

$$\hat{\pi}_{k\overline{l}} = \hat{\pi}_{\overline{kl}} + n_{k}\hat{\pi}_{\perp\overline{l}}, \quad \hat{\pi}_{\overline{kl}} = \frac{1}{3}\eta_{\overline{kl}}\hat{\pi} + \overset{\wedge}{\hat{\pi}}_{\overline{kl}} + \overset{\sim}{\hat{\pi}}_{\overline{kl}}. \tag{4.7}$$

In this expansion we identify the 0^{+} scalar $\hat{\pi}$, the antisymmetric 1^{+} vector $\overset{\wedge}{\hat{\pi}}_{\overline{kl}}$, the 1^{-} vector $\hat{\pi}_{\perp\overline{k}}$ and symmetric-traceless 2^{+} tensor $\overset{\sim}{\hat{\pi}}_{\overline{kl}}$. Applying this decomposition to (4.6) as a whole, we obtain four functions which, with the aid of (4.5), are simultaneously defined both in terms of canonical and non-canonical variables

$$\varphi \equiv J^{-1}\hat{\pi} = -4\hat{\beta}_{2}m_{\mathrm{p}}{}^{2}\eta^{\overline{kl}}\mathcal{T}^{\overline{k}}{}_{\overline{k}\perp}\ , \tag{4.8a}$$

$$\overset{\wedge}{\varphi}_{\overline{kl}} \equiv J^{-1}\overset{\wedge}{\hat{\pi}}_{\overline{kl}} - \frac{4}{3}(\hat{\beta}_{1} - \hat{\beta}_{3})m_{\mathrm{p}}{}^{2}\mathcal{T}_{\perp\overline{kl}} = -\frac{4}{3}(\hat{\beta}_{1} + 2\hat{\beta}_{3})m_{\mathrm{p}}{}^{2}\mathcal{T}_{[\overline{kl}]\perp}\ , \tag{4.8b}$$

$$\varphi_{\perp\overline{k}} \equiv J^{-1}\hat{\pi}_{\perp\overline{k}} - \frac{4}{3}(\hat{\beta}_{1} - \hat{\beta}_{2})m_{\mathrm{p}}{}^{2}\overset{\rightharpoonup}{\mathcal{T}}_{\overline{k}} = -\frac{4}{3}(2\hat{\beta}_{1} + \hat{\beta}_{2})m_{\mathrm{p}}{}^{2}\mathcal{T}_{\perp\overline{k}\perp}\ , \tag{4.8c}$$

$$\overset{\sim}{\varphi}_{\overline{kl}} \equiv J^{-1}\overset{\sim}{\hat{\pi}}_{\overline{kl}} = -4\hat{\beta}_{1}m_{\mathrm{p}}{}^{2}\mathcal{T}_{\langle\overline{kl}\rangle\perp}\ , \tag{4.8d}$$

where the vector and symmetric-traceless torsion are $\overset{\rightharpoonup}{\mathcal{T}}_{\overline{k}} \equiv \mathcal{T}^{\overline{i}}{}_{\overline{ki}}$ and $\mathcal{T}_{\langle\overline{kl}\rangle\perp} \equiv \mathcal{T}_{(\overline{kl})\perp} - \frac{1}{3}\eta_{\overline{kl}}\eta^{\overline{ij}}\mathcal{T}_{\overline{kl}\perp}$. In each case, *if* the combination of coupling constants appearing in the non-canonical RHS definition vanishes, the canonically defined function on the LHS becomes a primary 'if' constraint (PiC). An analogous construction is available for the second relation in (4.6), with the 18 remaining momenta decomposing as follows

$$\begin{gathered}\hat{\pi}_{kl\overline{m}} = \hat{\pi}_{\overline{klm}} + 2n_{[k}\hat{\pi}_{\perp\overline{l}]\overline{m}}, \quad \hat{\pi}_{\perp\overline{kl}} = \frac{1}{3}\eta_{\overline{kl}}\hat{\pi}_{\perp} + \overset{\wedge}{\hat{\pi}}_{\perp\overline{kl}} + \overset{\sim}{\hat{\pi}}_{\perp\overline{kl}}, \\ \hat{\pi}_{\overline{klm}} = \frac{1}{6}\epsilon_{\overline{klm}\perp}{}^{\mathrm{P}}\hat{\pi} + \overset{\rightharpoonup}{\hat{\pi}}_{[\overline{k}}\eta_{\overline{l}]\overline{m}} + \frac{4}{3}{}^{\mathrm{T}}\hat{\pi}_{\overline{klm}}.\end{gathered} \tag{4.9}$$

These are the 0^+ scalar $\hat{\pi}_\perp$, antisymmetric 1^+ vector $\overset{\wedge}{\hat{\pi}}_{\perp\overline{kl}}$, symmetric 2^+ tensor $\overset{\sim}{\hat{\pi}}_{\perp\overline{kl}}$, and then the 0^- pseudoscalar ${}^{\mathrm{P}}\hat{\pi}$, the 1^- vector $\overset{\rightharpoonup}{\hat{\pi}}_{\overline{k}}$ and 2^- tensor ${}^{\mathrm{T}}\hat{\pi}_{\overline{klm}}$. We will use (${}^{\mathrm{P}}\cdot$) to refer to the pseudoscalar part of general tensors, and (${}^{\mathrm{T}}\cdot_{\overline{klm}}$) to refer to the tensor part (with antisymmetry implicit in the *first* pair of indices, even if $\cdot_{\overline{klm}} \equiv \cdot_{\overline{k[lm]}}$). The six PiC functions from (4.6) are then

$$\varphi_\perp \equiv J^{-1}\hat{\pi}_\perp + 2\left(\hat{\alpha}_4 - \hat{\alpha}_6\right)\underline{\mathcal{R}} = 4\left(\hat{\alpha}_4 + \hat{\alpha}_6\right)\mathcal{R}_{\perp\perp}, \tag{4.10a}$$

$${}^{\mathrm{P}}\varphi \equiv J^{-1}\,{}^{\mathrm{P}}\hat{\pi} + 4\left(\hat{\alpha}_2 - \hat{\alpha}_3\right){}^{\mathrm{P}}\mathcal{R}_{\perp\circ} = -4\left(\hat{\alpha}_2 + \hat{\alpha}_3\right){}^{\mathrm{P}}\mathcal{R}_{\circ\perp}, \tag{4.10b}$$

$$\overset{\wedge}{\varphi}_{\perp\overline{kl}} \equiv J^{-1}\overset{\wedge}{\hat{\pi}}_{\perp\overline{kl}} - 4\left(\hat{\alpha}_2 - \hat{\alpha}_5\right)\underline{\mathcal{R}}_{[\overline{kl}]} = -4\left(\hat{\alpha}_2 + \hat{\alpha}_5\right)\mathcal{R}_{\perp[\overline{kl}]\perp}, \tag{4.10c}$$

$$\overset{\rightharpoonup}{\varphi}_{\overline{k}} \equiv J^{-1}\overset{\rightharpoonup}{\hat{\pi}}_{\overline{k}} + 4\left(\hat{\alpha}_4 - \hat{\alpha}_5\right)\mathcal{R}_{\perp\overline{k}} = -4\left(\hat{\alpha}_4 + \hat{\alpha}_5\right)\mathcal{R}_{\overline{k}\perp}, \tag{4.10d}$$

$$\overset{\sim}{\varphi}_{\perp\overline{kl}} \equiv J^{-1}\overset{\sim}{\hat{\pi}}_{\perp\overline{kl}} + 4\left(\hat{\alpha}_1 - \hat{\alpha}_4\right)\underline{\mathcal{R}}_{\langle\overline{kl}\rangle} = -4\left(\hat{\alpha}_1 + \hat{\alpha}_4\right)\mathcal{R}_{\perp\langle\overline{kl}\rangle\perp}, \tag{4.10e}$$

$${}^{\mathrm{T}}\varphi_{\overline{klm}} \equiv J^{-1}\,{}^{\mathrm{T}}\hat{\pi}_{\overline{klm}} - 4\left(\hat{\alpha}_1 - \hat{\alpha}_2\right){}^{\mathrm{T}}\mathcal{R}_{\perp\overline{klm}} = -4\left(\hat{\alpha}_1 + \hat{\alpha}_2\right){}^{\mathrm{T}}\mathcal{R}_{\overline{klm}\perp}, \tag{4.10f}$$

where we make further notational definitions ${}^{\mathrm{P}}\mathcal{R}_{\perp\circ} \equiv \epsilon^{\overline{ijk}\perp}\mathcal{R}_{\overline{ijk}\perp}$, ${}^{\mathrm{P}}\mathcal{R}_{\circ\perp} \equiv \epsilon^{\overline{ijk}\perp}\mathcal{R}_{\perp\overline{ijk}}$, $\underline{\mathcal{R}}_{\overline{kl}} \equiv \mathcal{R}^{\overline{i}}{}_{\overline{kli}}$, and $\underline{\mathcal{R}} \equiv \underline{\mathcal{R}}^{\overline{i}}{}_{\overline{i}}$. By this analysis, the possible occurence of primary constraints is systematically exhausted.

4.2.2 Secondary constraints and the Hamiltonian

In order to be consistent, a primary constraint should not have any velocity within the final mass shell, so its commutator with the *total* Hamiltonian $\mathcal{H}_{\mathrm{T}}$ should weakly vanish

$$\dot{\varphi}(x_1) \equiv \int \mathrm{d}^3x_2\Big\{\varphi(x_1), \mathcal{H}_{\mathrm{T}}(x_2)\Big\} \approx 0. \tag{4.11}$$

The Poisson bracket appearing in (4.11) and throughout this chapter is defined for general functionals $\mathcal{A}$ and $\mathcal{B}$ of the gravitational fields and their conjugate momenta, as set out in (4.2)

$$\Big\{\mathcal{A}, \mathcal{B}\Big\} \equiv \int \mathrm{d}^3x \left[\frac{\delta\mathcal{A}}{\delta b^i{}_\mu}\frac{\delta\mathcal{B}}{\delta\pi_i{}^\mu} + \frac{\delta\mathcal{A}}{\delta A^{ij}{}_\mu}\frac{\delta\mathcal{B}}{\delta\pi_{ij}{}^\mu} - \frac{\delta\mathcal{A}}{\delta\pi_i{}^\mu}\frac{\delta\mathcal{B}}{\delta b^i{}_\mu} - \frac{\delta\mathcal{A}}{\delta\pi_{ij}{}^\mu}\frac{\delta\mathcal{B}}{\delta A^{ij}{}_\mu}\right]. \tag{4.12}$$

The formula (4.12) may appear no more daunting than a commonplace action variation, but in practice $\mathcal{A}$ and $\mathcal{B}$ are frequently *local tensors* rather than nonlocal scalars. Locality (seen already in (4.11)) signifies that the underlying functionals contain Dirac distribuions, themselves subject to the total derivatives of the generalised Euler–Lagrange equations. The full ramifications of covariantly removing these Dirac gradients are detailed in Appendix C.9, while a certain flexibility will be assumed in the formula (4.12) in order to accommodate new gravitational D.o.F in Chapter 5. The total Hamiltonian is related to the *canonical* Hamiltonian $\mathcal{H}_{\mathrm{C}}$, the Legendre-transformed Lagrangian, by the constraints and their multiplier fields

$$\mathcal{H}_{\mathrm{T}} \equiv \mathcal{H}_{\mathrm{C}} + u^k{}_0\varphi_k{}^0 + \frac{1}{2}u^{ij}{}_0\varphi_{ij}{}^0 + (u\cdot\varphi), \tag{4.13}$$

where the last term schematically represents any PiCs which may arise. The canonical Hamiltonian may generally be collected into the insightful *Dirac* form [222, 223]

$$\mathcal{H}_{\mathrm{C}} \equiv N\mathcal{H}_\perp + N^\alpha\mathcal{H}_\alpha - \frac{1}{2}A^{ij}{}_0\mathcal{H}_{ij} + \partial_\alpha\mathcal{D}^\alpha, \tag{4.14}$$

i.e. as a linear function of the non-physical fields up to a surface term. The remaining functions which appear in (4.14) are defined as follows

$$\mathcal{H}_\perp \equiv \hat{\pi}_i{}^{\overline{k}}\mathcal{T}^i{}_{\perp\overline{k}} + \frac{1}{2}\hat{\pi}_{ij}{}^{\overline{k}}\mathcal{R}^{ij}{}_{\perp\overline{k}} - JL - n^k D_\alpha \pi_k{}^\alpha, \tag{4.15a}$$

$$\mathcal{H}_\alpha \equiv \pi_i{}^\beta T^i{}_{\alpha\beta} + \frac{1}{2}\pi_{ij}{}^\beta R^{ij}{}_{\alpha\beta} - b^k{}_\alpha D_\beta \pi_k{}^\beta, \tag{4.15b}$$

$$\mathcal{H}_{ij} \equiv 2\pi_{[i}{}^\alpha b_{j]\alpha} + D_\alpha \pi_{ij}{}^\alpha, \tag{4.15c}$$

$$\mathcal{D}^\alpha \equiv b^i{}_0 \pi_i{}^\alpha + \frac{1}{2}A^{ij}{}_0 \pi_{ij}{}^\alpha. \tag{4.15d}$$

It is clear from (4.11) and the Dirac form (4.14) that the consistency of (4.3) invokes 10 'sure' *secondary* first class (sSFC) constriants

$$\mathcal{H}_\perp \approx 0, \quad \mathcal{H}_\alpha \approx 0, \quad \mathcal{H}_{ij} \approx 0. \tag{4.16}$$

It is important to note that while the sSFCs in (4.16) are always enforced, it does not always follow that 2×10 D.o.F are removed from the theory, as is the case with the sPFCs in (4.3). The functions involved are quite complicated, and may degenerately express a reduced number of FCs, or FCs which only appear at deeper levels in the consistency chain. Indeed, while this is very rare in the literature, we will find that it occurs for all eight novel theories, as a consequence of vanishing mass parameters.

The linear and rotational super-momenta $\mathcal{H}_\alpha$ and $\mathcal{H}_{ij}$ are kinematic generators which do not impinge on the dynamics. Thus, in the evaluation of (4.11), it is sufficient to work purely with the super-Hamiltonian $\mathcal{H}_\perp$, which is, at length, expanded out using Eqs. (4.4) and (4.5) to give

$$\begin{aligned}
\mathcal{H}_\perp &= m_{\mathrm{p}}{}^2 J \sum_{I=1}^{3} \hat{\beta}_I \Big[4\mathcal{T}^i{}_{\perp\overline{k}}\, {}^I\mathcal{P}_i{}^{\perp\overline{k}}{}_j{}^{\perp\overline{l}}\mathcal{T}^j{}_{\perp\overline{l}} - \mathcal{T}^i{}_{\overline{mk}}\, {}^I\mathcal{P}_i{}^{\overline{mk}}{}_j{}^{\overline{nl}}\mathcal{T}^j{}_{\overline{nl}}\Big] \\
&\quad + J\sum_{I=1}^{6}\hat{\alpha}_I\Big[4\mathcal{R}^{ip}{}_{\perp\overline{k}}\, {}^I\mathcal{P}_{ip}{}^{\perp\overline{k}}{}_{jq}{}^{\perp\overline{l}}\mathcal{R}^{jq}{}_{\perp\overline{l}} - \mathcal{R}^{ip}{}_{\overline{mk}}\, {}^I\mathcal{P}_{ip}{}^{\overline{mk}}{}_{jq}{}^{\overline{nl}}\mathcal{R}^{jq}{}_{\overline{nl}}\Big] - n^k D_\alpha \pi_k{}^\alpha \\
&= \frac{J}{16}\Bigg[\frac{2\varphi^2}{3\hat{\beta}_2 m_{\mathrm{p}}{}^2} + \frac{6\hat{\varphi}_{\overline{kl}}\hat{\varphi}^{\overline{kl}}}{(\hat{\beta}_1+2\hat{\beta}_3)m_{\mathrm{p}}{}^2} + \frac{6\varphi_{\perp\overline{k}}\varphi^{\perp\overline{k}}}{(2\hat{\beta}_1+\hat{\beta}_2)m_{\mathrm{p}}{}^2} + \frac{2\tilde{\varphi}_{\overline{kl}}\tilde{\varphi}^{\overline{kl}}}{\hat{\beta}_1 m_{\mathrm{p}}{}^2} + \frac{2\varphi_\perp{}^2}{3(\hat{\alpha}_4+\hat{\alpha}_6)} + \frac{{}^{\mathrm{P}}\varphi^2}{6(\hat{\alpha}_2+\hat{\alpha}_3)} \\
&\quad + \frac{2\hat{\varphi}_{\perp\overline{kl}}\hat{\varphi}^{\perp\overline{kl}}}{\hat{\alpha}_2+\hat{\alpha}_5} + \frac{\vec{\varphi}_{\overline{k}}\vec{\varphi}^{\overline{k}}}{\hat{\alpha}_4+\hat{\alpha}_5} + \frac{2\tilde{\varphi}_{\perp\overline{kl}}\tilde{\varphi}^{\perp\overline{kl}}}{\hat{\alpha}_1+\hat{\alpha}_4} + \frac{16\,{}^{\mathrm{T}}\varphi_{\overline{klm}}\,{}^{\mathrm{T}}\varphi^{\overline{klm}}}{9(\hat{\alpha}_1+\hat{\alpha}_2)}\Bigg] + J\Bigg[\frac{1}{3}(2\hat{\beta}_1+\hat{\beta}_3)m_{\mathrm{p}}{}^2\mathcal{T}_{\perp\overline{kl}}\,\mathcal{T}^{\perp\overline{kl}} \\
&\quad + \frac{1}{3}(\hat{\beta}_1+2\hat{\beta}_2)m_{\mathrm{p}}{}^2\vec{\mathcal{T}}_{\overline{k}}\vec{\mathcal{T}}^{\overline{k}} - \frac{1}{6}\hat{\beta}_3 m_{\mathrm{p}}{}^2\,{}^{\mathrm{P}}\mathcal{T}^2 + \frac{16}{9}\hat{\beta}_1 m_{\mathrm{p}}{}^2\,{}^{\mathrm{T}}\mathcal{T}_{\overline{klm}}\,{}^{\mathrm{T}}\mathcal{T}^{\overline{klm}} + \frac{1}{6}(\hat{\alpha}_4+\hat{\alpha}_6)\underline{\mathcal{R}}^2 \\
&\quad - \frac{1}{6}(\hat{\alpha}_2+\hat{\alpha}_3)\,{}^{\mathrm{P}}\mathcal{R}_{\perp\circ}{}^2 + 2(\hat{\alpha}_2+\hat{\alpha}_5)\underline{\mathcal{R}}_{[\overline{kl}]}\underline{\mathcal{R}}^{[\overline{kl}]} + (\hat{\alpha}_4+\hat{\alpha}_5)\mathcal{R}_{\perp\overline{k}}\mathcal{R}^{\perp\overline{k}} + 2(\hat{\alpha}_1+\hat{\alpha}_4)\underline{\mathcal{R}}_{\langle\overline{kl}\rangle}\underline{\mathcal{R}}^{\langle\overline{kl}\rangle} \\
&\quad + \frac{16}{9}(\hat{\alpha}_1+\hat{\alpha}_2)\,{}^{\mathrm{T}}\mathcal{R}_{\perp\overline{klm}}\,{}^{\mathrm{T}}\mathcal{R}^{\perp\overline{klm}}\Bigg] - n^k D_\alpha \pi_k{}^\alpha.
\end{aligned} \tag{4.17}$$

To arrive at the second equality in (4.17), the non-canonical 'perpendicular' field strengths appearing in the first equality are canonicalised at length by the dual PiC definitions in Eqs. (4.8a) to (4.8d) and Eqs. (4.10a) to (4.10f), resulting in terms quadratic in the PiC functions, and in the canonical 'parallel' field strengths. The resulting expression is quite lengthy, but can be simplified for any given theory by safely eliminting those PiC functions which become constraints. The signs of the remaining

Table 4.2 Spin-parity sectors and associated PiCs, along with their kinetic and mass parameters. For completeness, we include the ${m_{\mathrm{p}}}^2\mathcal{R}$ term, mediated by $\hat{\alpha}_0$. Coupling translations provided in Eqs. (B.23j) to (B.23l).

J^P	PiC	Kinetic	PiC–PiC masses	PiC–SiC masses
0^+	φ	$\hat{\beta}_2$	$\{{\varphi_\perp}^\flat, \varphi^\flat\} \sim (2\hat{\alpha}_0 + \hat{\beta}_2)\eta$	$\{\varphi^\flat, \chi^\flat\} \sim (2\hat{\alpha}_0 + \hat{\beta}_2)\eta$
	$\varphi_\perp$	$\hat{\alpha}_4 + \hat{\alpha}_6$		$\{{\varphi_\perp}^\flat, {\chi^\flat}_\perp\} \sim \hat{\alpha}_0(2\hat{\alpha}_0 + \hat{\beta}_2)\eta$
0^-	${}^{\mathrm{P}}\varphi$	$\hat{\alpha}_2 + \hat{\alpha}_3$	(lonely)	$\{{}^{\mathrm{P}}\varphi^\flat, {}^{\mathrm{P}}\chi^\flat\} \sim (\hat{\alpha}_0 + 2\hat{\beta}_3)\eta$
1^+	$\overset{\wedge}{\varphi}_{\overline{kl}}$	$\hat{\beta}_1 + 2\hat{\beta}_3$	$\{{\overset{\wedge}{\varphi}^\flat}_{\perp\overline{ij}}, {\overset{\wedge}{\varphi}^\flat}_{\overline{lm}}\} \sim (\hat{\alpha}_0 + 2\hat{\beta}_3)\eta$	$\{{\overset{\wedge}{\varphi}^\flat}_{\overline{ij}}, {\overset{\wedge}{\chi}^\flat}_{\overline{lm}}\} \sim (\hat{\alpha}_0 + 2\hat{\beta}_3)\eta$
	$\overset{\wedge}{\varphi}_{\perp\overline{kl}}$	$\hat{\alpha}_2 + \hat{\alpha}_5$		$\{{\overset{\wedge}{\varphi}^\flat}_{\perp\overline{ij}}, {\overset{\wedge}{\chi}^\flat}_{\perp\overline{lm}}\} \sim (\hat{\alpha}_0 + 2\hat{\beta}_3)(\hat{\alpha}_0 - \hat{\beta}_1)\eta$
1^-	$\varphi_{\perp\overline{k}}$	$2\hat{\beta}_1 + \hat{\beta}_2$	$\{{\vec{\varphi}^\flat}_{\overline{i}}, {\varphi^\flat}_{\perp\overline{l}}\} \sim (2\hat{\alpha}_0 + \hat{\beta}_2)\eta$	$\{{\varphi^\flat}_{\perp\overline{i}}, {\chi^\flat}_{\perp\overline{l}}\} \sim (2\hat{\alpha}_0 + \hat{\beta}_2)\eta$
	$\vec{\varphi}_{\overline{k}}$	$\hat{\alpha}_4 + \hat{\alpha}_5$		$\{{\vec{\varphi}^\flat}_{\overline{i}}, {\vec{\chi}^\flat}_{\overline{l}}\} \sim (2\hat{\alpha}_0 + \hat{\beta}_2)(\hat{\alpha}_0 - \hat{\beta}_1)\eta$
2^+	$\tilde{\varphi}_{\overline{kl}}$	$\hat{\beta}_1$	$\{{\tilde{\varphi}^\flat}_{\perp\overline{ij}}, {\tilde{\varphi}^\flat}_{\overline{lm}}\} \sim (\hat{\alpha}_0 - \hat{\beta}_1)\eta$	$\{{\tilde{\varphi}^\flat}_{\perp\overline{ij}}, {\tilde{\chi}^\flat}_{\perp\overline{lm}}\} \sim (\hat{\alpha}_0 - \hat{\beta}_1)\eta$
	$\tilde{\varphi}_{\perp\overline{kl}}$	$\hat{\alpha}_1 + \hat{\alpha}_4$		$\{{\tilde{\varphi}^\flat}_{\perp\overline{ij}}, {\tilde{\chi}^\flat}_{\perp\overline{lm}}\} \sim \hat{\alpha}_0(\hat{\alpha}_0 - \hat{\beta}_1)\eta$
2^-	${}^{\mathrm{T}}\varphi_{\overline{klm}}$	$\hat{\alpha}_1 + \hat{\alpha}_2$	(lonely)	$\{{}^{\mathrm{T}}{\varphi^\flat}_{\overline{ijk}}, {}^{\mathrm{T}}{\chi^\flat}_{\overline{lmn}}\} \sim (\hat{\alpha}_0 - \hat{\beta}_1)\eta$

quadratic PiC terms are then instrumental in the identification of unconstrained ghosts, since the PiC functions are schematically of the form $\varphi \sim \pi + \mathcal{R}$ or $\varphi \sim \pi + \mathcal{T}$.

The consistency of the PiCs is less straightforward. Generally, the PiCs may be FC or SC within their own mass shell. In the case that a PiC is FC, (4.11) provides a secondary if-constraint (SiC). Possibly, the PiC and SiC do not commute; in that case both become SC within the new mass shell and the consistency of the SiC allows a multiplier to be determined

$$\dot{\chi}(x_1) \equiv \int \mathrm{d}^3 x_2 \Big(N \Big\{ \chi(x_1), \mathcal{H}_\perp(x_2) \Big\} + u \cdot \Big\{ \chi(x_1), \varphi(x_2) \Big\} \Big) \approx 0. \tag{4.18}$$

Otherwise, a tertiary if-constraint (TiC) may be found, and the process continues until the constraint chain from the PiC is absorbed by another chain, or by the sSFCs. In the case that a PiC is already SC within the PiC mass shell, its chain terminates immediately and two multipliers are determined. We note that occasionally, a constraint may be encountered at some deep level which retroactively terminates the chain at a shallower point. Only once the algoritm has terminated is it safe to categorise the if-constraints as FC or SC.

In the linearised theory [224], the analysis is greatly simplified by an understanding of the mass spectrum [225]. Only the $\mathcal{O}(1)$ parts of the PiC commutators contribute to the evaluation of the multipliers. Such commutators are possible only between pairs of PiCs which belong to the same SO(3) irrep, and which are known as *conjugate pairs* [226]. Conjugate PiCs will fail to commute in the linear theory only when their common mass parameter is non-vanishing. In this case, if only one PiC in a pair is present, it will still fail to commute with the SiC that maintains its consistency. Particularly, the rotational ${}^{\mathrm{P}}\varphi$ and ${}^{\mathrm{T}}\varphi_{\overline{klm}}$ have no SO(3) counterparts in the translational sector, and are conjugate with their secondaries ${}^{\mathrm{P}}\chi$ and ${}^{\mathrm{T}}\chi_{\overline{klm}}$ a priori. In the case of *vanishing* mass parameters, the PiCs are FC, and a new gauge symmetry is invoked. The PiCs belonging to various SO(3) irreps, along with their kinetic parameters and linearised mass parameters are listed in Table 4.2.

Note that up to this point, our discussion has been fully general, and lays the theoretical foundations and conventions for the forthcoming series. The evaluation of Poisson brackets is made tedious by the dependence of various quantities on the translational gauge field, as illustrated by the following useful identities

$$\frac{\partial n_l}{\partial b^k{}_\mu} \equiv -n_k h_{\bar{l}}{}^\mu, \quad \frac{\partial h_l{}^\nu}{\partial b^k{}_\mu} \equiv -h_k{}^\nu h_l{}^\mu, \quad \frac{\partial b}{\partial b^k{}_\nu} \equiv b h_k{}^\nu, \quad \frac{\partial J}{\partial b^k{}_\nu} \equiv J h_{\bar{k}}{}^\nu, \quad \frac{\partial N}{\partial b^k{}_\nu} \equiv N n_k h_\perp{}^\nu. \tag{4.19}$$

As a crude measure to simplify the calculations, we artificially restrict our analysis in this chapter to theories whose PiCs among Eqs. (4.10a) to (4.10f) do not depend on $\mathcal{R}^{ij}{}_{\overline{kl}}$. It must be emphasised that this does not (to our knowledge) translate into any useful restriction on the physics. Of the 58 novel theories in [152, 153], eight satisfy our criterion: Case 3, Case 17, Case 20, Case 24, Case ^{*5}25, Case ^{*6}26, Case 28 and Case 32. For most of these cases, we are fortunate that the remaining PiCs among Eqs. (4.8a) to (4.8d) also do not depend on $\mathcal{T}^i{}_{\overline{kl}}$. Case 3 and Case 17 are exceptions to this rule. We detail in Table 4.1 our prior understanding of these theories, as encoded by the saturated graviton and roton propagators, linearised on Minkowski spacetime in the absence of matter. Aside from having torsion-dependent PiCs, Case 3 and Case 17 are particularly interesting as candidate theories of gravity, as they contain two massless D.o.F with power in the 2^+ part of the propagator – we will return to this point in Section 4.4.

From our discussion in Section 4.2.2, we see that the constraint structure of the theory depends partially on the commutators between the PiCs, which form the *primary Poisson matrix* (PPM). In Sections 4.3 and 4.4 we will use the structure of the nonlinear PPM as a proxy for the health of each theory.

4.3 Massive-only results

4.3.1 Case ^{*6}26

Conveniently, the PiCs of the massive theories depend on neither $\mathcal{T}^i{}_{\overline{jk}}$ nor $\mathcal{R}^{ij}{}_{\overline{kl}}$, so we will have schematically $\varphi \sim \pi$ for both translational and rotational sectors. By substituting the definition of Case ^{*6}26 from Table 4.1 into (4.17) and (4.13), the total Hamiltonian is seen to take the form

$$\mathcal{H}_{\mathrm{T}} = \frac{b}{96}\left(\frac{18\hat{\varphi}_{\overline{kl}}\hat{\varphi}^{\overline{kl}}}{\hat{\beta}_3} - \frac{{}^{\mathrm{P}}\varphi^2}{\hat{\alpha}_3}\right) + \text{fields}, \tag{4.20}$$

where we include only the part quadratic in the momenta. The remaining eight PiC functions that do not appear in (4.20) are primarily constrained, and give rise to the following nonvanishing commutators within the PiC shell

$$\left\{\varphi_{\perp\bar{i}}, \varphi_{\perp\bar{l}}\right\} \approx \frac{2}{J^2}\hat{\hat{\pi}}_{\overline{il}}\delta^3, \tag{4.21a}$$

$$\left\{\varphi_{\perp\bar{i}}, \hat{\varphi}_{\perp\overline{lm}}\right\} \approx -\frac{1}{6J^2}\epsilon_{\overline{ilm}\perp}{}^{\mathrm{P}}\hat{\pi}\delta^3, \tag{4.21b}$$

$$\left\{\tilde{\varphi}_{\overline{ij}}, \tilde{\varphi}_{\overline{lm}}\right\} \approx \frac{1}{J^2}\left[\eta_{\bar{i}(\bar{l}|}\hat{\hat{\pi}}_{\bar{j}|\bar{m})} + \eta_{\bar{j}(\bar{l}|}\hat{\hat{\pi}}_{\bar{i}|\bar{m})}\right]\delta^3, \tag{4.21c}$$

$$\left\{\tilde{\varphi}_{\overline{ij}}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\} \approx \frac{1}{24J^2}\left[2\eta_{(\bar{i}|\bar{n}}\epsilon_{|\bar{j})\overline{lm}\perp} - \eta_{(\bar{i}|\bar{l}}\epsilon_{|\bar{j})\overline{mn}\perp} + \eta_{(\bar{i}|\bar{m}}\epsilon_{|\bar{j})\overline{ln}\perp}\right]{}^{\mathrm{P}}\hat{\pi}\delta^3, \tag{4.21d}$$

where δ^3 represents the equal-time Dirac function. The nonlinear PPM of Case $^{*6}26$ is then written:

$$\left[\mathsf{M}^{\left(\text{Case }^{*6}26\right)}_{\propto}\right] \approx \begin{array}{c|cccc|cccc|c}
 & \varphi & \varphi_{\perp\overline{k}} & \tilde{\varphi}_{\overline{kl}} & \varphi_{\perp} & \hat{\varphi}_{\perp\overline{kl}} & \vec{\varphi}_{\overline{k}} & \tilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klm}} & \\
\hline
\varphi & \cdot & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \cdot & \cdot & 1 \\
\varphi_{\perp\overline{k}} & \cdot & \hat{\pi} & \cdot & \cdot & \hat{\pi}! & \boxed{\cdot} & \cdot & \cdot & 3 \\
\tilde{\varphi}_{\overline{kl}} & \cdot & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \boxed{\cdot} & \hat{\pi}! & 5 \\
\varphi_{\perp} & \boxed{\cdot} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\hline
\hat{\varphi}_{\perp\overline{kl}} & \cdot & \hat{\pi}! & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\vec{\varphi}_{\overline{k}} & \cdot & \boxed{\cdot} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
\tilde{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\
{}^{\mathrm{T}}\varphi_{\overline{klm}} & \cdot & \cdot & \hat{\pi}! & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\
\hline
 & 1 & 3 & 5 & 1 & 3 & 3 & 5 & 5 &
\end{array} \tag{4.22}$$

The elements of the matrix schematically represent the nonlinear Poisson brackets between the PiCs. The PiCs are labelled, along with their multiplicities, at the edges of the PPM. They are arranged so as to divide the matrix into translational and rotational blocks, separated by ($+$). All brackets are restricted to the PiC shell. Commuting PiCs are dentoted by ($\cdot$). Non-commuting PiCs denoted as ($\hat{\pi}$) are strictly linear combinations of $\hat{\pi}_{ij}{}^{\overline{k}}$ and $\hat{\pi}_i{}^{\overline{k}}$ as detailed in Eqs. (C.15a) to (4.21d). Generally, these expressions can be quite lengthy, so henceforth we confine them to Appendix C.3. Commutators depending on momenta which (as we shall shortly show) propagate in the final linear theory are denoted by ($\hat{\pi}!$). These are significant as they are presumed to persist even when the full nonlinear Dirac–Bergmann algorithm is terminated, except possibly on any strongly coupled spacetimes which might be found away from Minkowski spacetime. Constant terms only arise in brackets between conjugate PiCs ($\boxed{\cdot}$), and then only if both PiCs have non-vanishing mass parameters. Since all the PiC mass parameters vanish in Case $^{*6}26$, no constant terms can arise. The linearised theory is sensitive only to these constant terms, but we see from (4.22) that the conjugate PiCs also commute in the nonlinear Case $^{*6}26$.

Let us now consider the consistency of the PiCs, and implement the Dirac–Bergmann algorithm for the linearised theory [227–229]. Within the PiC shell, we encounter the following SiCs

$$\chi^{\flat}{}_{\perp\overline{k}} \approx -2\eta^{\flat\overline{ml}}\mathcal{D}^{\flat}{}_{\overline{m}}\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}}, \quad \hat{\chi}^{\flat}{}_{\perp\overline{kl}} \approx 2\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}} - \frac{1}{6}\epsilon^{\flat}{}_{\overline{klm}\perp}\eta^{\flat\overline{mn}}\mathcal{D}^{\flat}{}_{\overline{n}}{}^{\mathrm{P}}\hat{\pi}^{\flat}, \tag{4.23}$$

where quantities linearised on the Minkowski background[2] are denoted with ($\flat$). Also within this shell, we find $\mathcal{H}^{\flat}{}_{\perp}$ and $\mathcal{H}^{\flat}{}_{\perp\overline{k}}$ already vanish weakly, while the linear super-momentum and vector part of the rotational super-momentum give further sSFCs

$$\mathcal{H}^{\flat}{}_{\alpha} \approx -h^{\flat}{}_{\alpha}{}^{\overline{j}}\eta^{\flat\overline{kl}}\mathcal{D}^{\flat}{}_{\overline{k}}\hat{\hat{\pi}}^{\flat}{}_{\overline{jl}}, \quad \mathcal{H}^{\flat}{}_{\overline{kl}} \approx 2\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}} - \frac{1}{6}\epsilon^{\flat}{}_{\overline{klm}\perp}\eta^{\flat\overline{mn}}\mathcal{D}^{\flat}{}_{\overline{n}}{}^{\mathrm{P}}\hat{\pi}^{\flat}. \tag{4.24}$$

The SiCs clearly vanish in the sSFC sub-shell, terminating the algorithm immediately. We find that $\mathcal{H}^{\flat}{}_{\alpha}$ is already implied by $\mathcal{H}^{\flat}{}_{\overline{kl}}$, which constitutes a total of three sSFCs. The PiCs are all FC, and the total number of iPFCs can be read off from (4.22). Recalling also the 10 sPFCs, and counting all FCs twice, we find that there is only one propagating D.o.F, as expected from Table 4.1

$$1 = \frac{1}{2}\big(80 - 2\times 10[\text{sPFC}] - 2\times 3[\text{sSFC}] - 2\times(1+3+5+1+3+3+5+5)[\text{iPFC}]\big). \tag{4.25}$$

[2]Note also that we use the linearised gauge covariant derivative $\mathcal{D}^{\flat}{}_{\overline{i}}$, even to replace the nonlinear coordinate derivative $h^{\flat}{}_{\overline{i}}{}^{\mu}\partial_{\mu}$.

So, what is this D.o.F? We know that there are 26 undetermined multipliers, to match each of the iPFCs. Generically, this makes it very difficult to make sense of the E.o.M. However, we can make an educated guess by noticing that the functions $\overset{\wedge}{\varphi}_{\overline{kl}}$ and ${}^{\mathrm{P}}\varphi$ are not PiCs and in the end, it turns out to be the 0^- tordion which is propagating. An application of (4.11) allows us to find the velocity of the pseudoscalar part of the torsion

$$ {}^{\mathrm{P}}\dot{\mathcal{T}}^{\flat} \approx -\frac{1}{2\hat{\alpha}_3}{}^{\mathrm{P}}\hat{\pi}^{\flat} - \frac{3}{4\hat{\beta}_3 {m_{\mathrm{p}}}^2}\epsilon^{\flat\overline{jkl}\perp}\mathcal{D}^{\flat}{}_{\overline{j}}\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\overline{kl}}. \tag{4.26} $$

Conveniently, we see that this quantity makes no reference to undetermined multipliers in the final shell. Moreover, the same can be said of the acceleration

$$ {}^{\mathrm{P}}\ddot{\mathcal{T}}^{\flat} \approx -\eta^{\flat\overline{jk}}\mathcal{D}^{\flat}{}_{\overline{j}}\mathcal{D}^{\flat}{}_{\overline{k}}{}^{\mathrm{P}}\mathcal{T}^{\flat} - \frac{4\hat{\beta}_3}{\hat{\alpha}_3}{m_{\mathrm{p}}}^2\,{}^{\mathrm{P}}\mathcal{T}^{\flat}, \tag{4.27} $$

which clearly describes a particle of mass

$$ m \equiv 2\sqrt{\frac{|\hat{\beta}_3|}{|\hat{\alpha}_3|}}m_{\mathrm{p}}, \tag{4.28} $$

if $\hat{\beta}_3/\hat{\alpha}_3 > 0$. The unitarity conditions in Table 4.1 can now be decoded. The condition $\hat{\alpha}_3 < 0$ clearly wards off a 0^- ghost by inspection of (4.20), whereas $\hat{\beta}_3 < 0$ then prevents the 0^- from becoming tachyonic.

In the nonlinear theory, the PPM is no longer empty as shown in (4.22). We anticipate that the emergent commutators will ultimately result in a fundamentally different particle spectrum. Particularly, we see from Eqs. (4.21c) and (4.21d) that $\varphi_{\perp\overline{k}}$, $\tilde{\varphi}_{\overline{kl}}$, $\overset{\wedge}{\varphi}_{\perp\overline{kl}}$ and ${}^{\mathrm{T}}\varphi_{\overline{klm}}$ are all demoted from iPFCs to iPSCs so long as 0^- is activated. Possibly, 0^- becomes strongly coupled on some other privileged surface within the final shell, but since the converse is unlikely to be true we conclude that an iPFC *generally* becomes an iPSC in the nonlinear theory. According to Dirac's conjecture, the FCs are associated with gauge symmetries. More correctly, every PFC can be used to construct a nontrivial gauge generator using the Castellani algorithm [230]. We therefore expect that a generator is generally broken.

To see one way in which this might affect the outcome, imagine that none of the sSFCs are degenerate in the full nonlinear theory, but that they still encode the iSFCs (which therefore need not appear in the final count). The nonlinear theory would then be expected to propagate two D.o.F

$$ 2 \overset{\text{(e.g.)}}{=} \frac{1}{2}\big(80 - 2\times 10[\text{sPFC}] - 2\times 10[\text{sSFC}] - 2\times(1+1+3+5)[\text{iPFC}] - (3+5+3+5)[\text{iPSC}]\big), \tag{4.29} $$

suggesting that somehow one D.o.F from the 1^+ sector (i.e. the only J^P other than 0^- which is not primarily constrained), is generally activated, but becomes strongly coupled on Minkowski spacetime. It is not clear what this would look like, and we emphasise that the specific scenario in (4.29) is unlikely to be the one which is realised. The full picture can only be revealed by performing the nonlinear Dirac–Bergmann analysis, beginning from (4.22). Following treatments of simpler cases of PGT$^{\mathrm{q}+}$ in [169], we will not go this far. However, we think it likely that any activation of the 1^+ sector will damage the unitarity of the theory, since we see from (4.20) that $\overset{\wedge}{\hat{\pi}}_{\overline{kl}}\overset{\wedge}{\hat{\pi}}{}^{\overline{kl}}$ has a negative contribution to the energy, by the same condition $\hat{\beta}_3 < 0$ that upholds the unitarity of the 0^- mode. For further discussion of the ‘positive energy test’, we direct the reader to Appendix C.2.

Finally, (4.22) may also indicate that the nonlinear theory violates causality. We refer to the test based on the tachyonic shock in the nonlinear Proca theory [220], and which was also implemented in [169], whereby the PPM rank is required not to depend on the values of the fields and their momenta. The motivation for this requirement is as follows. It is easy to see from (4.11) that the multipliers ${}^{A}u$ and ${}^{B}u$ of a pair of PiCs ${}^{A}\varphi$ and ${}^{B}\varphi$ can be determined in the case that $\{{}^{A}\varphi, {}^{B}\varphi\} \not\approx 0$ on the final shell. Moreover, ${}^{A}u$ will be nonvanishing if $\{\mathcal{H}_{\mathrm{C}}, {}^{B}\varphi\} \not\approx 0$. Imagine that a dynamical trajectory intersected a surface Σ on which $\{{}^{A}\varphi, {}^{B}\varphi\} \to 0$. The multiplier ${}^{A}u$ had better not have any physical interpretation in that case, since it would diverge[3]. Unfortunately in the case of $\mathrm{PGT}^{\mathrm{q}+}$, the multipliers can be written in terms of the non-canonical velocities[4] through the dual definitions of the PiC functions in Eqs. (4.8a) to (4.8d) and Eqs. (4.10a) to (4.10f). The interpretation is then that a tachyonic excitation develops on the approach to Σ. In the case at hand, the nonlinear PPM in (4.22) is populated by momenta, and the linearised PPM is empty. Thus, Minkowski spacetime is just such a surface Σ. More generally, when the linearised PPM is populated by constant mass parameters, the requirement becomes that the nonlinear PPM pseudodeterminant should be positive-definite within the final shell.

4.3.2 Case 28

Since Case 28 has fewer PiCs than Case ^{*6}26, the kinetic part of the Hamiltonian is more extensive

$$\mathcal{H}_{\mathrm{T}} = \frac{b}{96}\left(\frac{6\left(\overrightarrow{\varphi}_{\overline{k}}\overrightarrow{\varphi}^{\overline{k}} + 2\hat{\varphi}_{\perp\overline{kl}}\hat{\varphi}^{\perp\overline{kl}}\right)}{\hat{\alpha}_5} + \frac{18\hat{\varphi}_{\overline{kl}}\hat{\varphi}^{\overline{kl}}}{\hat{\beta}_3} - \frac{{}^{\mathrm{P}}\varphi^2}{\hat{\alpha}_3}\right) + \text{fields}, \tag{4.30}$$

while the PPM has fewer dimensions

$$\left[\mathsf{M}^{(\text{Case 28})}\right] \approx \begin{array}{r|cccccc|l} & \varphi & \varphi_{\perp\overline{k}} & \tilde{\varphi}_{\overline{kl}} & \varphi_{\perp} & \tilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klm}} & \\ \hline \varphi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \varphi_{\perp\overline{k}} & \cdot & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi} & \hat{\pi} & 3 \\ \tilde{\varphi}_{\overline{kl}} & \cdot & \cdot & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi}! & 5 \\ \varphi_{\perp} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\ \tilde{\varphi}_{\perp\overline{kl}} & \cdot & \hat{\pi} & \hat{\pi} & \cdot & \cdot & \cdot & 5 \\ {}^{\mathrm{T}}\varphi_{\overline{klm}} & \cdot & \hat{\pi} & \hat{\pi}! & \cdot & \cdot & \cdot & 5 \\ \hline & 1 & 3 & 5 & 1 & 5 & 5 & \end{array} \tag{4.31}$$

Within the PiC shell, we find that $\varphi^{\flat}$ and $\tilde{\varphi}^{\flat}{}_{\overline{kl}}$ already weakly vanish, leaving the following SiCs

$$\begin{gathered} \chi^{\flat}{}_{\perp\overline{k}} \approx -2\eta^{\flat\overline{ml}}\mathcal{D}^{\flat}{}_{\overline{m}}\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}}, \quad \chi^{\flat}{}_{\perp} \approx -\eta^{\flat\overline{ml}}\mathcal{D}^{\flat}{}_{\overline{m}}\overrightarrow{\hat{\pi}}^{\flat}{}_{\overline{l}}, \quad \tilde{\chi}^{\flat}{}_{\perp\overline{kl}} \approx \frac{1}{2}\mathcal{D}^{\flat}{}_{\langle\overline{k}}\overrightarrow{\hat{\pi}}^{\flat}{}_{\overline{l}\rangle}, \\ {}^{\mathrm{T}}\chi^{\flat}{}_{\overline{klm}} \approx \frac{1}{2}\mathcal{D}^{\flat}{}_{\overline{m}}\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{kl}} + \frac{1}{2}\mathcal{D}^{\flat}{}_{[\overline{l}}\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{k}]\overline{m}} + \frac{3}{4}\eta^{\flat}{}_{\overline{m}[\overline{k}|}\eta^{\flat\overline{ij}}\mathcal{D}^{\flat}{}_{\overline{i}}\hat{\hat{\pi}}^{\flat}{}_{\perp|\overline{l}]\overline{j}}, \end{gathered} \tag{4.32}$$

[3]The problem is somewhat analogous to one of strong coupling. If the prefactor to the kinetic term of a field vanishes (i.e. its mass becomes infinite) on some Σ, the Heisenberg principle suggests that quantum fluctuations will diverge on the approach to Σ.

[4]We note a caveat here, that this interpretation is strictly true for theories with nonvanishing mass parameters; more careful investigation of the multiplier interpretation may be warranted for the cases at hand.

which do not give rise to any TiCs. Also within the PiC shell, the following sSFCs appear

$$\begin{aligned}&\mathcal{H}^{\flat}{}_{\alpha} \approx -h^{\flat}{}_{\alpha}{}^{\overline{j}}\eta^{\flat\overline{kl}}\mathcal{D}^{\flat}{}_{\overline{k}}\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\overline{jl}}, \quad \mathcal{H}^{\flat}{}_{\overline{kl}} \approx 2\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\overline{kl}} - \frac{1}{6}\epsilon^{\flat}{}_{\overline{klm}\perp}\eta^{\flat\overline{mn}}\mathcal{D}^{\flat}{}_{\overline{n}}{}^{\mathrm{P}}\hat{\pi}^{\flat} + \mathcal{D}^{\flat}{}_{[\overline{k}}\overset{\rightharpoonup}{\hat{\pi}}{}^{\flat}{}_{\overline{l}]}, \\ &\mathcal{H}^{\flat}{}_{\perp\overline{k}} \approx \eta^{\flat\overline{jl}}\mathcal{D}^{\flat}{}_{\overline{j}}\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\perp\overline{kl}}.\end{aligned} \tag{4.33}$$

In this case it is easiest to restrict to sub-shells using the SiCs and sSFCs simultaneously. We first note that $\mathcal{H}^{\flat}{}_{\perp\overline{k}}$ restricts $\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\perp\overline{kl}}$ to be solenoidal, dual to the gradient of a scalar, and thus eliminates two D.o.F. The remaining D.o.F is eliminated by ${}^{\mathrm{T}}\chi^{\flat}{}_{\overline{klm}}$. Similarly, $\chi^{\flat}{}_{\perp}$ restricts $\overset{\rightharpoonup}{\hat{\pi}}{}^{\flat}{}_{\overline{k}}$ to a solenoidal axial vector, removing one D.o.F. A further D.o.F is removed by substituting $\mathcal{H}^{\flat}{}_{\overline{kl}}$ into $\mathcal{H}^{\flat}{}_{\alpha}$, and a final D.o.F is removed by $\widetilde{\chi}^{\flat}{}_{\perp\overline{kl}}$. Separately, $\mathcal{H}^{\flat}{}_{\overline{kl}}$ removes three D.o.Fs. All the PiCs and SiCs are FC, and one D.o.F remains, as expected from Table 4.1

$$\begin{aligned}1 = \frac{1}{2}\big(&80 - 2\times 10[\mathrm{sPFC}] - 2\times(1+3+2)[\mathrm{sSFC}] - 2\times(1+3+5+1+5+5)[\mathrm{iPFC}] \\ &- 2\times(1+1+1)[\mathrm{iSFC}]\big).\end{aligned} \tag{4.34}$$

As with Case ${}^{*6}26$, the no-ghost condition $\hat{\alpha}_3 < 0$ protects the 0^- mode in (4.30). However, we note that the linearly-propagating ${}^{\mathrm{P}}\hat{\pi}$ again emerges at the nonlinear level in (4.31), so that a linear gauge symmetry is broken and (4.34) is not valid sufficiently far from Minkowski spacetime. Whether or not an increase in the propagating D.o.F results in a ghost is not so clear in Case 28 as it was in Case ${}^{*6}26$. From (4.30), we see that an activation of $\overset{\wedge}{\hat{\pi}}{}_{\overline{kl}}$ would endanger nonlinear unitarity by the linear no-tachyon condition $\hat{\beta}_3 < 0$. However, if either of the vector tordions $\overset{\rightharpoonup}{\hat{\pi}}{}_{\overline{k}}$ or $\overset{\wedge}{\hat{\pi}}{}_{\perp\overline{kl}}$ were to propagate, positive-definite contributions to $\mathcal{H}_{\mathrm{T}}$ could be ensured by respectively fixing $\hat{\alpha}_5 < 0$ or $\hat{\alpha}_5 > 0$, since $\hat{\alpha}_5$ does not serve to shore up the linearised unitarity. The key point here, as discussed in Appendix C.2, is that with our 'West Coast' signature every contraction on parallel indices introduces a factor of -1. Therefore, if *both* vector tordions propate in the nonlinear theory, it would seem that negative kinetic energy contributions to $\mathcal{H}_{\mathrm{T}}$ are unavoidable. Whatever the status of ghosts, we observe that the nonlinear PPM has field-dependent rank.

4.3.3 Case ${}^{*5}25$

The structure of Case ${}^{*5}25$ has many similarities with that of Case 28. The Hamiltonian takes the form

$$\mathcal{H}_{\mathrm{T}} = \frac{b}{96}\left(\frac{4\left(\varphi^2 + 9\varphi_{\perp\overline{k}}\varphi^{\perp\overline{k}}\right)}{\hat{\beta}_2} + \frac{18\overset{\wedge}{\varphi}_{\overline{kl}}\overset{\wedge}{\varphi}{}^{\overline{kl}}}{\hat{\beta}_3} - \frac{{}^{\mathrm{P}}\varphi^2}{\hat{\alpha}_3}\right) + \text{fields}, \tag{4.35}$$

while the nonlinear PPM is more sparsely populated:

$$\left[\mathsf{M}^{\left(\text{Case } {}^{*5}25\right)}_{\propto}\right] \approx \begin{array}{r|cccccc|l} & \widetilde{\varphi}_{\overline{kl}} & \varphi_{\perp} & \overset{\wedge}{\varphi}_{\perp\overline{kl}} & \overset{\rightharpoonup}{\varphi}_{\overline{k}} & \widetilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klm}} & \\ \widetilde{\varphi}_{\overline{kl}} & \hat{\pi} & \cdot & \cdot & \cdot & \boxed{\cdot} & \hat{\pi}! & 5 \\ \varphi_{\perp} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \overset{\wedge}{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\ \overset{\rightharpoonup}{\varphi}_{\overline{k}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\ \widetilde{\varphi}_{\perp\overline{kl}} & \boxed{\cdot} & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\ {}^{\mathrm{T}}\varphi_{\overline{klm}} & \hat{\pi}! & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\ \hline & 5 & 1 & 3 & 3 & 5 & 5 & \end{array} \tag{4.36}$$

Within the PiC shell, we have

$$\tilde{\chi}^{\flat}{}_{\overline{kl}} \approx -\mathcal{D}^{\flat}{}_{\langle\overline{k}}\hat{\pi}^{\flat}{}_{\perp\overline{l}\rangle} \quad \chi^{\flat}{}_{\perp} \approx \hat{\pi}^{\flat}, \quad \hat{\chi}^{\flat}{}_{\perp\overline{kl}} \approx 2\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}} - \frac{1}{6}\epsilon^{\flat}{}_{\overline{klm}\perp}\eta^{\flat\overline{mn}}\mathcal{D}^{\flat}{}_{\overline{n}}{}^{\mathrm{P}}\hat{\pi}^{\flat}, \quad \vec{\chi}^{\flat}{}_{\overline{k}} \approx 2\hat{\pi}^{\flat}{}_{\perp\overline{k}}, \tag{4.37}$$

and this time, all 10 sSFCs persist in the PiC shell

$$\begin{aligned} &\mathcal{H}^{\flat}{}_{\perp} \approx -\eta^{\flat\overline{kl}}\mathcal{D}^{\flat}{}_{\overline{k}}\hat{\pi}^{\flat}{}_{\perp\overline{l}}, \quad \mathcal{H}^{\flat}{}_{\alpha} \approx -\frac{1}{3}h^{\flat}{}_{\alpha}{}^{\overline{k}}\mathcal{D}^{\flat}{}_{\overline{k}}\hat{\pi}^{\flat} - h^{\flat}{}_{\alpha}{}^{\overline{k}}\eta^{\flat\overline{jl}}\mathcal{D}^{\flat}{}_{\overline{j}}\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}}, \\ &\mathcal{H}^{\flat}{}_{\overline{kl}} \approx 2\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}} - \frac{1}{6}\epsilon^{\flat}{}_{\overline{klm}\perp}\eta^{\flat\overline{mn}}\mathcal{D}^{\flat}{}_{\overline{n}}{}^{\mathrm{P}}\hat{\pi}^{\flat}, \quad \mathcal{H}^{\flat}{}_{\perp\overline{k}} \approx \hat{\pi}^{\flat}{}_{\perp\overline{k}}. \end{aligned} \tag{4.38}$$

We find that $\mathcal{H}^{\flat}{}_{\perp\overline{k}}$ and $\mathcal{H}^{\flat}{}_{\overline{kl}}$ each remove three D.o.F, while $\chi^{\flat}{}_{\perp}$ removes one D.o.F; the remaining sSFCs and SiCs are then implied, and the PiCs and SiCs are FC. Once again, one D.o.F remains as expected from Table 4.1

$$\begin{aligned} 1 = \frac{1}{2}\big(&80 - 2\times 10[\text{sPFC}] - 2\times(3+3)[\text{sSFC}] - 2\times(5+1+3+3+5+5)[\text{iPFC}] \\ &- 2\times 1[\text{iSFC}]\big). \end{aligned} \tag{4.39}$$

The discussion now proceeds in much the same way as with Case 28, since PiC commutators linear in the propagating ${}^{\mathrm{P}}\hat{\pi}$ emerge away from Minkowski spacetime. This time, it is the tetrad momenta $\hat{\pi}$ and $\hat{\pi}_{\perp\overline{k}}$ which introduce extra uncertainty regarding ghosts. If only one of these momenta becomes activated, $\hat{\beta}_2$ may be used to ensure it has a positive contribution to $\mathcal{H}_{\mathrm{T}}$. Again, the nonlinear PPM rank is field-dependent.

4.3.4 Case 24

Case 24 has only 16 PiCs, the fewest out of all the cases we consider. The kinetic part of the Hamiltonian is proportionally more complicated

$$\mathcal{H}_{\mathrm{T}} = \frac{b}{96}\left(\frac{6\big(\vec{\varphi}_{\overline{k}}\vec{\varphi}^{\overline{k}2} + 2\hat{\varphi}_{\perp\overline{kl}}\hat{\varphi}^{\perp\overline{kl}}\big)}{\hat{\alpha}_5} + \frac{18\hat{\varphi}_{\overline{kl}}\hat{\varphi}^{\overline{kl}}}{\hat{\beta}_3} + \frac{4\big(\varphi^2 + 9\varphi_{\perp\overline{k}}\varphi^{\perp\overline{k}}\big)}{\hat{\beta}_2} - \frac{{}^{\mathrm{P}}\varphi^2}{\hat{\alpha}_3}\right) + \text{fields}, \tag{4.40}$$

while the PPM is extremely small:

$$\left[\mathsf{M}^{(\text{Case 24})}_{\propto}\right] \approx \begin{array}{c|cccc|c} & \tilde{\varphi}_{\overline{kl}} & \varphi_{\perp} & \tilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klm}} & \\ \hline \tilde{\varphi}_{\overline{kl}} & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi}! & 5 \\ \varphi_{\perp} & \cdot & \cdot & \cdot & \cdot & 1 \\ \tilde{\varphi}_{\perp\overline{kl}} & \hat{\pi} & \cdot & \cdot & \cdot & 5 \\ {}^{\mathrm{T}}\varphi_{\overline{klm}} & \hat{\pi}! & \cdot & \cdot & \cdot & 5 \\ \hline & 5 & 1 & 5 & 5 & \end{array} \tag{4.41}$$

Within the PiC shell, we have the following SiCs

$$\begin{aligned} &\tilde{\chi}^{\flat}{}_{\overline{kl}} \approx -\mathcal{D}^{\flat}{}_{\langle\overline{k}}\hat{\pi}^{\flat}{}_{\perp\overline{l}\rangle} \quad \chi^{\flat}{}_{\perp} \approx \hat{\pi}^{\flat} - \eta^{\flat\overline{kl}}\mathcal{D}^{\flat}{}_{\overline{k}}\vec{\hat{\pi}}^{\flat}{}_{\overline{l}}, \quad \tilde{\chi}^{\flat}{}_{\perp\overline{kl}} \approx \frac{1}{2}\mathcal{D}^{\flat}{}_{\langle\overline{k}}\vec{\hat{\pi}}^{\flat}{}_{\overline{l}\rangle}, \\ &{}^{\mathrm{T}}\chi^{\flat}{}_{\overline{klm}} \approx \frac{1}{2}\mathcal{D}^{\flat}{}_{\overline{m}}\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{kl}} + \frac{1}{2}\mathcal{D}^{\flat}{}_{[\overline{l}}\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{k}]\overline{m}} + \frac{3}{4}\eta^{\flat}{}_{\overline{m}[\overline{k}|}\eta^{\flat\overline{ij}}\mathcal{D}^{\flat}{}_{\overline{i}}\hat{\hat{\pi}}^{\flat}{}_{\perp|\overline{l}]\overline{j}}, \end{aligned} \tag{4.42}$$

and the following sPFCs

$$\begin{aligned}&\mathcal{H}^{\flat}{}_{\perp} \approx -\eta^{\flat\overline{kl}}\mathcal{D}^{\flat}{}_{\overline{k}}\hat{\pi}^{\flat}{}_{\perp\overline{l}} \quad \mathcal{H}^{\flat}{}_{\alpha} \approx -\frac{1}{3}h^{\flat}{}_{\alpha}{}^{\overline{k}}\mathcal{D}^{\flat}{}_{\overline{k}}\hat{\pi}^{\flat} - h^{\flat}{}_{\alpha}{}^{\overline{k}}\eta^{\flat\overline{jl}}\mathcal{D}^{\flat}{}_{\overline{j}}\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}},\\ &\mathcal{H}^{\flat}{}_{\overline{kl}} \approx 2\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}} - \frac{1}{6}\epsilon^{\flat}{}_{\overline{klm}\perp}\eta^{\flat\overline{mn}}\mathcal{D}^{\flat}{}_{\overline{n}}{}^{\mathrm{P}}\hat{\pi}^{\flat} + \mathcal{D}^{\flat}{}_{[\overline{k}}\vec{\hat{\pi}}^{\flat}{}_{\overline{l}]}, \quad \mathcal{H}^{\flat}{}_{\perp\overline{k}} \approx \hat{\pi}^{\flat}{}_{\perp\overline{k}} + \eta^{\flat\overline{jl}}\mathcal{D}^{\flat}{}_{\overline{j}}\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{kl}}.\end{aligned} \tag{4.43}$$

It is clear from the PiC shell that Case 24 has much in common with Case 28, and again we will implement the SiCs and sSFCs simultaneously. Firstly, we find that ${}^{\mathrm{T}}\chi^{\flat}{}_{\overline{klm}}$ constitutes an overdetermined system in $\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{kl}}$, which vanishes and takes with it three D.o.F. Consequently, from $\mathcal{H}^{\flat}{}_{\perp\overline{k}}$, we see that $\hat{\pi}^{\flat}{}_{\perp\overline{k}}$ must vanish along with another three D.o.F, such that $\mathcal{H}^{\flat}{}_{\perp}$ and $\tilde{\chi}^{\flat}{}_{\overline{kl}}$ vanish automatically. Similarly, $\tilde{\chi}^{\flat}{}_{\perp\overline{kl}}$ is an overdetermined system in $\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{kl}}$, which vanishes with another three D.o.F; $\chi^{\flat}{}_{\perp}$ then causes $\hat{\pi}^{\flat}$ to vanish with one D.o.F. As before, one D.o.F propagates in accordance with Table 4.1

$$\begin{aligned}1 = \frac{1}{2}\big(&80 - 2\times 10[\text{sPFC}] - 2\times(3+3)[\text{sSFC}] - 2\times(5+1+5+5)[\text{iPFC}]\\ &- 2\times(1+3+3)[\text{iSFC}]\big).\end{aligned} \tag{4.44}$$

It is clear from (4.40) that any inference of the nonlinear unitarity will just combine the discussions of Case 28 and Case ^{*5}25, while the PPM rank is again field dependent.

4.3.5 Case 32

For the first time, we encounter non-vanishing mass parameters between the PiCs, specifically in $\hat{\varphi}^{\flat}{}_{\perp\overline{kl}}$ and ${}^{\mathrm{T}}\varphi^{\flat}{}_{\overline{klm}}$. We anticipate the nonvanishing commutators even at the linear level $\left\{\hat{\varphi}^{\flat}{}_{\perp\overline{kl}}, \hat{\chi}^{\flat}{}_{\perp\overline{ij}}\right\} \approx \mathcal{O}(1)$ – noting that the natural conjugate $\hat{\varphi}^{\flat}{}_{\overline{kl}}$ is not a PiC – and $\left\{{}^{\mathrm{T}}\varphi^{\flat}{}_{\overline{klm}}, {}^{\mathrm{T}}\chi^{\flat}{}_{\overline{ijn}}\right\} \approx \mathcal{O}(1)$. These PiCs and SiCs will be SC, allowing for the determination of their multipliers. The kinetic part of the Hamiltonian is

$$\mathcal{H}_{\mathrm{T}} = \frac{b}{96}\left(\frac{6\big(3\varphi_{\perp\overline{k}}\varphi^{\perp\overline{k}} + 2\tilde{\varphi}_{\overline{kl}}\tilde{\varphi}^{\overline{kl}}\big)}{\hat{\beta}_1} + \frac{36\hat{\varphi}_{\overline{kl}}\hat{\varphi}^{\overline{kl}}}{\hat{\beta}_1 + 2\hat{\beta}_3} - \frac{{}^{\mathrm{P}}\varphi^2}{\hat{\alpha}_3}\right) + \text{fields}. \tag{4.45}$$

In the PPM, we label the PiCs associated with nonvanishing mass parameters by $(\downarrow)$, producing:

$$\left[\mathsf{M}^{(\text{Case 32})}_{\propto}\right] \approx \begin{array}{r|cccccc|l} & & & \downarrow & & & \downarrow & \\ & \varphi & \varphi_{\perp} & \hat{\varphi}_{\perp\overline{kl}} & \vec{\varphi}_{\overline{k}} & \tilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klm}} & \\ \hline \varphi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1\\ \varphi_{\perp} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1\\ \rightarrow\ \hat{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3\\ \vec{\varphi}_{\overline{k}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3\\ \tilde{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5\\ \rightarrow\ {}^{\mathrm{T}}\varphi_{\overline{klm}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5\\ \hline & 1 & 1 & 3 & 3 & 5 & 5 & \end{array} \tag{4.46}$$

Thus the PPM of this theory is remarkable, since it remains empty even in the nonlinear regime. Within the PiC shell, we find the following SiCs

$$\begin{aligned} &\chi^\flat \approx -\eta^{\flat\overline{kl}}\mathcal{D}^\flat{}_{\overline{k}}\hat{\pi}^\flat{}_{\perp\overline{l}}, \\ &\overset{\wedge}{\chi}{}^\flat{}_{\perp\overline{kl}} \approx -\frac{\hat{\beta}_1+2\hat{\beta}_3}{\hat{\beta}_1-\hat{\beta}_3}\overset{\wedge}{\hat{\pi}}{}^\flat{}_{\overline{kl}} - \frac{1}{6}\epsilon^\flat{}_{\overline{klm}\perp}\eta^{\flat\overline{mn}}\mathcal{D}^\flat{}_{\overline{n}}{}^{\mathrm{P}}\hat{\pi}^\flat + \frac{9\hat{\beta}_1\hat{\beta}_3}{(\hat{\beta}_1-\hat{\beta}_3)(\hat{\beta}_1+2\hat{\beta}_3)}\overset{\wedge}{\varphi}{}^\flat{}_{\overline{kl}}, \\ &\overset{\rightharpoonup}{\chi}{}^\flat{}_{\overline{k}} \approx -\hat{\pi}^\flat{}_{\perp\overline{k}}, \quad \overset{\sim}{\chi}{}^\flat{}_{\perp\overline{kl}} \approx \overset{\sim}{\hat{\pi}}{}^\flat{}_{\overline{kl}}, \quad {}^{\mathrm{T}}\chi^\flat{}_{\overline{klm}} \approx 4\hat{\beta}_1 m_{\mathrm{p}}{}^2\,{}^{\mathrm{T}}\mathcal{T}^\flat{}_{\overline{klm}}\,. \end{aligned} \tag{4.47}$$

Note the appearance of field strengths, specifically the torsion in $\overset{\wedge}{\varphi}{}^\flat{}_{\overline{kl}}$ and ${}^{\mathrm{T}}\mathcal{T}^\flat{}_{\overline{klm}}$. Whilst these somewhat complicate the analysis, they naturally appear with the mass parameters. We also mark the first apparent instance of a TiC accompanying $\overset{\sim}{\chi}{}^\flat{}_{\perp\overline{kl}}$. Using the notation $\zeta \equiv \dot{\chi}$, this may be written as

$$\overset{\sim}{\zeta}{}^\flat{}_{\perp\overline{kl}} \approx \frac{4}{3}\eta^{\flat\overline{ij}}\mathcal{D}^\flat{}_{\overline{i}}{}^{\mathrm{T}}\chi^\flat{}_{\langle\overline{k}|\overline{j}|\overline{l}\rangle}, \tag{4.48}$$

which then vanishes in the SiC shell. The PiC shell contains the following sSFCs:

$$\begin{aligned} &\mathcal{H}^\flat{}_\perp \approx -\eta^{\flat\overline{kl}}\mathcal{D}^\flat{}_{\overline{k}}\hat{\pi}^\flat{}_{\perp\overline{l}} \quad \mathcal{H}^\flat{}_\alpha \approx -h^\flat{}_\alpha{}^{\overline{k}}\eta^{\flat\overline{jl}}\mathcal{D}^\flat{}_{\overline{j}}\overset{\wedge}{\hat{\pi}}{}^\flat{}_{\overline{kl}} - h^\flat{}_\alpha{}^{\overline{k}}\eta^{\flat\overline{jl}}\mathcal{D}^\flat{}_{\overline{j}}\overset{\sim}{\hat{\pi}}{}^\flat{}_{\overline{kl}}, \\ &\mathcal{H}^\flat{}_{\overline{kl}} \approx 2\overset{\wedge}{\hat{\pi}}{}^\flat{}_{\overline{kl}} - \frac{1}{6}\epsilon^\flat{}_{\overline{klm}\perp}\eta^{\flat\overline{mn}}\mathcal{D}^\flat{}_{\overline{n}}{}^{\mathrm{P}}\hat{\pi}^\flat, \quad \mathcal{H}^\flat{}_{\perp\overline{k}} \approx \hat{\pi}^\flat{}_{\perp\overline{k}}. \end{aligned} \tag{4.49}$$

Since two of the PiC chains are known to be self-terminating, the algorithm concludes quite quickly. As with Case ^{*5}25, $\mathcal{H}^\flat{}_{\perp\overline{k}}$ and $\mathcal{H}^\flat{}_{\overline{kl}}$ each eliminate three D.o.F. Another five D.o.F are then removed by $\overset{\sim}{\chi}{}^\flat{}_{\perp\overline{kl}}$, with the remaining SiCs and sSFCs automatically satisfied. The one remaining D.o.F is again expected from Table 4.1

$$\begin{aligned} 1 = \frac{1}{2}\big(&80 - 2\times 10[\text{sPFC}] - 2\times(3+3)[\text{sSFC}] - 2\times(1+1+3+5)[\text{iPFC}] - (3+5)[\text{iPSC}] \\ &- 2\times 5[\text{iSFC}] - (3+5)[\text{iSSC}]\big). \end{aligned} \tag{4.50}$$

On the whole, the outlook for Case 32 appears more promising than for the previous cases, because the PPM retains its empty structure (and rank) when passing to the nonlinear regime. This is just the first hurdle, as the full nonlinear algorithm would still be required to determine whether further fields become activated. The implications of field activaton are slightly relaxed, compared to Case ^{*5}25 or Case 28. The linear tachyon condition $\hat{\beta}_3 < 0$ need not imply that a propagating $\overset{\wedge}{\pi}_{\overline{kl}}$ contributes negative kinetic energy if $\hat{\beta}_1 + 2\hat{\beta}_3 > 0$. This can be realised even if $\overset{\sim}{\pi}_{\overline{kl}}$ is simultaneously activated. However for positive kinetic energy it seems $\hat{\pi}_{\perp\overline{k}}$ must be activated on its own or not at all, since $\hat{\beta}_1 < 0$ would then be required.

4.3.6 Case 20

The analysis of Case 20 is quite similar to Case 32. Mass parameters again accompany the PiCs, and we expect $\overset{\wedge}{\varphi}{}^\flat{}_{\perp\overline{kl}}$, $\overset{\rightharpoonup}{\varphi}{}^\flat{}_{\overline{k}}$ and ${}^{\mathrm{T}}\varphi^\flat{}_{\overline{klm}}$ to not commute with their respective SiCs on the final shell. The kinetic part of the Hamiltonian is

$$\mathcal{H}_{\mathrm{T}} = \frac{b}{96}\left(\frac{4\varphi^2}{\hat{\beta}_2} + \frac{12\overset{\sim}{\varphi}_{\overline{kl}}\overset{\sim}{\varphi}{}^{\overline{kl}}}{\hat{\beta}_1} + \frac{36\overset{\wedge}{\varphi}_{\overline{kl}}\overset{\wedge}{\varphi}{}^{\overline{kl}}}{\hat{\beta}_1+2\hat{\beta}_3} + \frac{36\varphi_{\perp\overline{k}}\varphi^{\perp\overline{k}}}{2\hat{\beta}_1+\hat{\beta}_2} - \frac{{}^{\mathrm{P}}\varphi^2}{\hat{\alpha}_3}\right) + \text{fields}, \tag{4.51}$$

and once again the PPM is empty both before and after linearisation:

$$\left[\mathsf{M}^{(\text{Case 20})}_{\propto}\right] \approx \begin{array}{c|ccccc|c} & \varphi_{\perp} & \overset{\downarrow}{\hat{\varphi}_{\perp\overline{kl}}} & \overset{\downarrow}{\vec{\varphi}_{\overline{k}}} & \tilde{\varphi}_{\perp\overline{kl}} & \overset{\downarrow}{{}^{\mathrm{T}}\varphi_{\overline{klm}}} & \\ \hline \varphi_{\perp} & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \rightarrow \hat{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\ \rightarrow \vec{\varphi}_{\overline{k}} & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\ \tilde{\varphi}_{\perp\overline{kl}} & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\ \rightarrow {}^{\mathrm{T}}\varphi_{\overline{klm}} & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\ \hline & 1 & 3 & 3 & 5 & 5 & \end{array} \tag{4.52}$$

Within the PiC shell, we first the following SiCs

$$\begin{aligned} \chi^{\flat}{}_{\perp} &\approx \hat{\pi}^{\flat}, \\ \hat{\chi}^{\flat}{}_{\perp\overline{kl}} &\approx -\frac{\hat{\beta}_1+2\hat{\beta}_3}{\hat{\beta}_1-\hat{\beta}_3}\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}} - \frac{1}{6}\epsilon^{\flat}{}_{\overline{klm}\perp}\eta^{\flat\overline{mn}}\mathcal{D}^{\flat}{}_{\overline{n}}{}^{\mathrm{P}}\hat{\pi}^{\flat} + \frac{9\hat{\beta}_1\hat{\beta}_3}{(\hat{\beta}_1-\hat{\beta}_3)(\hat{\beta}_1+2\hat{\beta}_3)}\hat{\varphi}^{\flat}{}_{\overline{kl}}, \\ \vec{\chi}^{\flat}{}_{\overline{k}} &\approx -\frac{\hat{\beta}_1+2\hat{\beta}_2}{\hat{\beta}_1-\hat{\beta}_2}\hat{\pi}^{\flat}{}_{\perp\overline{k}} + \frac{9\hat{\beta}_1\hat{\beta}_2}{(\hat{\beta}_1-\hat{\beta}_2)(2\hat{\beta}_1+\hat{\beta}_2)}\varphi^{\flat}{}_{\perp\overline{k}}, \quad \tilde{\chi}^{\flat}{}_{\perp\overline{kl}} \approx \tilde{\hat{\pi}}^{\flat}{}_{\overline{kl}}, \\ {}^{\mathrm{T}}\chi^{\flat}{}_{\overline{klm}} &\approx 4\hat{\beta}_1 m_{\mathrm{p}}{}^{2}\,{}^{\mathrm{T}}\mathcal{T}^{\flat}{}_{\overline{klm}}\,. \end{aligned} \tag{4.53}$$

This time, two TiCs appear, but upon rearranging both may eventually be written in terms of the iSSCs, and are therefore satisfied automatically

$$\zeta^{\flat}{}_{\perp} \approx \eta^{\flat\overline{ij}}\mathcal{D}^{\flat}{}_{\overline{i}}\vec{\chi}^{\flat}{}_{\overline{j}}, \quad \tilde{\zeta}^{\flat}{}_{\perp\overline{kl}} \approx \frac{4}{3}\eta^{\flat\overline{ij}}\mathcal{D}^{\flat}{}_{\overline{i}}{}^{\mathrm{T}}\chi^{\flat}{}_{\langle\overline{k}|\overline{j}|\overline{l}\rangle} - \frac{1}{2}\mathcal{D}^{\flat}{}_{\langle\overline{k}}\vec{\chi}^{\flat}{}_{\overline{l}\rangle}. \tag{4.54}$$

The sSFC content in the PiC shell is largely the same as that of Case 32, with the only difference marked in the linear super-momentum

$$\mathcal{H}^{\flat}{}_{\alpha} \approx -\frac{1}{3}h^{\flat}{}_{\alpha}{}^{\overline{k}}\mathcal{D}^{\flat}{}_{\overline{k}}\hat{\pi}^{\flat} - h^{\flat}{}_{\alpha}{}^{\overline{k}}\eta^{\flat\overline{jl}}\mathcal{D}^{\flat}{}_{\overline{j}}\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}} - h^{\flat}{}_{\alpha}{}^{\overline{k}}\eta^{\flat\overline{jl}}\mathcal{D}^{\flat}{}_{\overline{j}}\tilde{\hat{\pi}}^{\flat}{}_{\overline{kl}}. \tag{4.55}$$

Aided by the additional conjugate pair of constraints, the algorithm terminates even faster than with Case 32: we see that one and five D.o.F are removed by each of $\chi^{\flat}{}_{\perp}$ and $\tilde{\varphi}^{\flat}{}_{\perp\overline{kl}}$. As before, the one propagating D.o.F is confirmed from Table 4.1

$$\begin{aligned} 1 = \frac{1}{2}\big(&80 - 2\times 10[\text{sPFC}] - 2\times(3+3)[\text{sSFC}] - 2\times(1+5)[\text{iPFC}] - (3+3+5)[\text{iPSC}] \\ &- 2\times(1+5)[\text{iSFC}] - (3+3+5)[\text{iSSC}]\big). \end{aligned} \tag{4.56}$$

If positive kinetic energy is a requirement, it seems that the momenta $\hat{\pi}$ and $\hat{\pi}_{\perp\overline{k}}$ in combination with one or more of $\tilde{\hat{\pi}}_{\overline{kl}}$ or $\hat{\hat{\pi}}_{\overline{kl}}$, should not all be activated at the same time.

4.4 Massless results

4.4.1 Case 17

Two theories in Table 4.1 – Case 17 and Case 3 – admit a pair of massless modes according to the linearised analysis. Beginning with Case 17, we find the Hamiltonian to have the structure

$$\mathcal{H}_{\mathrm{T}} = \frac{b}{32}\left(\frac{2\left(\vec{\varphi}_{\overline{k}}\vec{\varphi}^{\overline{k}} + 2\hat{\varphi}_{\perp\overline{kl}}\hat{\varphi}^{\perp\overline{kl}}\right)}{\hat{\alpha}_5} - \frac{3\varphi_{\perp\overline{k}}\varphi^{\perp\overline{k}} + 2\tilde{\varphi}_{\overline{kl}}\tilde{\varphi}^{\overline{kl}}}{\hat{\beta}_3}\right) + \text{fields}, \tag{4.57}$$

As mentioned in Section 4.2.2, the evaluation of the PPM is complicated by the appearance of torsion in PiC $\hat{\varphi}_{\overline{kl}}$ belonging to the translational sector. In general, commutators between field strengths generate derivatives of the Dirac function. In many cases, these derivatives either happen to cancel, or they may be discarded up to a surface term within the PiC shell. In any case, we find that the full nonlinear PPM can be written purely in terms of the parallel momenta as before:

$$\left[\mathsf{M}^{(\text{Case 17})}_{\propto}\right] \approx \begin{array}{cc|cccccc|c}
 & & \downarrow & & \downarrow & & & \downarrow & \\
 & & \varphi & \hat{\varphi}_{\overline{kl}} & \varphi_{\perp} & {}^{\mathrm{P}}\varphi & \tilde{\varphi}_{\perp\overline{kl}} & {}^{\mathrm{T}}\varphi_{\overline{klm}} & \\
\hline
 & \varphi & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\
\rightarrow & \hat{\varphi}_{\overline{kl}} & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi} & \hat{\pi} & \hat{\pi} & 3 \\
 & \varphi_{\perp} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\
\rightarrow & {}^{\mathrm{P}}\varphi & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 1 \\
 & \tilde{\varphi}_{\perp\overline{kl}} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 5 \\
\rightarrow & {}^{\mathrm{T}}\varphi_{\overline{klm}} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & \cdot & 5 \\
\hline
 & & 1 & 3 & 1 & 1 & 5 & 5 &
\end{array} \tag{4.58}$$

Due to the appearance of mass parameters, we will expect $\hat{\varphi}^{\flat}{}_{\overline{kl}}$, ${}^{\mathrm{P}}\varphi^{\flat}$ and ${}^{\mathrm{T}}\varphi^{\flat}{}_{\overline{klm}}$ not to commute with their SiCs in the final shell. Within the PiC shell, we find the following SiCs

$$\begin{aligned}
\chi^{\flat} &\approx -\eta^{\flat\overline{kl}}\mathcal{D}^{\flat}{}_{\overline{k}}\hat{\pi}^{\flat}{}_{\perp\overline{l}}, \quad \hat{\chi}^{\flat}{}_{\overline{kl}} \approx -\frac{2\hat{\beta}_3}{\hat{\alpha}_5}m_{\mathrm{p}}{}^2\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{kl}} - \mathcal{D}^{\flat}{}_{[\overline{k}}\hat{\pi}^{\flat}{}_{\perp\overline{l}]} - 8\hat{\beta}_3 m_{\mathrm{p}}{}^2\mathcal{D}^{\flat}{}_{[\overline{k}}\vec{\mathcal{T}}^{\flat}{}_{\overline{l}]}, \\
\chi^{\flat}{}_{\perp} &\approx -\eta^{\flat\overline{kl}}\mathcal{D}^{\flat}{}_{\overline{k}}\vec{\hat{\pi}}^{\flat}{}_{\overline{l}}, \quad {}^{\mathrm{P}}\chi^{\flat} \approx 2\epsilon^{\flat\overline{jkl}\perp}\mathcal{D}^{\flat}{}_{\overline{j}}\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{kl}} + 8\hat{\beta}_3 m_{\mathrm{p}}{}^{2\,\mathrm{P}}\mathcal{T}^{\flat}, \quad \tilde{\chi}^{\flat}{}_{\perp\overline{kl}} \approx \tilde{\hat{\pi}}^{\flat}{}_{\overline{kl}} + \frac{1}{2}\mathcal{D}^{\flat}{}_{\langle\overline{k}}\vec{\hat{\pi}}^{\flat}{}_{\overline{l}\rangle}, \\
{}^{\mathrm{T}}\chi^{\flat}{}_{\overline{klm}} &\approx \frac{1}{2}\mathcal{D}^{\flat}{}_{\overline{m}}\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{kl}} + \frac{1}{2}\mathcal{D}^{\flat}{}_{[\overline{l}}\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{k}]\overline{m}} + \frac{3}{4}\eta^{\flat}{}_{\overline{m}[\overline{k}|}\eta^{\flat\overline{ij}}\mathcal{D}^{\flat}{}_{\overline{i}}\hat{\hat{\pi}}^{\flat}{}_{\perp|\overline{l}]\overline{j}} - 8\hat{\beta}_3 m_{\mathrm{p}}{}^{2\,\mathrm{T}}\mathcal{T}^{\flat}{}_{\overline{klm}}\,.
\end{aligned} \tag{4.59}$$

Among these, we note that a TiC accompanies $\tilde{\chi}^{\flat}{}_{\perp\overline{kl}}$, but may be expressed in terms of ${}^{\mathrm{T}}\chi^{\flat}{}_{\overline{klm}}$ by precisely (4.48). Within the PiC shell, the sSFCs are

$$\begin{aligned}
\mathcal{H}^{\flat}{}_{\perp} &\approx -\eta^{\flat\overline{kl}}\mathcal{D}^{\flat}{}_{\overline{k}}\hat{\pi}^{\flat}{}_{\perp\overline{l}}, \quad \mathcal{H}^{\flat}{}_{\alpha} \approx -h^{\flat}{}_{\alpha}{}^{\overline{k}}\eta^{\flat\overline{jl}}\mathcal{D}^{\flat}{}_{\overline{j}}\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}} - h^{\flat}{}_{\alpha}{}^{\overline{k}}\eta^{\flat\overline{jl}}\mathcal{D}^{\flat}{}_{\overline{j}}\tilde{\hat{\pi}}^{\flat}{}_{\overline{kl}}, \\
\mathcal{H}^{\flat}{}_{\overline{kl}} &\approx 2\hat{\hat{\pi}}^{\flat}{}_{\overline{kl}} + \mathcal{D}^{\flat}{}_{[\overline{k}}\vec{\hat{\pi}}^{\flat}{}_{\overline{l}]}, \quad \mathcal{H}^{\flat}{}_{\perp\overline{k}} \approx \hat{\pi}^{\flat}{}_{\perp\overline{k}} + \eta^{\flat\overline{jl}}\mathcal{D}^{\flat}{}_{\overline{j}}\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{kl}}.
\end{aligned} \tag{4.60}$$

The conjugate pairs together eliminate six, two and 10 D.o.F before terminating. As with Case 28, $\vec{\hat{\pi}}^{\flat}{}_{\overline{k}}$ becomes solenoidal due to $\chi^{\flat}{}_{\perp}$ and loses one D.o.F, while three D.o.F are lost by each of $\mathcal{H}^{\flat}{}_{\overline{kl}}$ and $\mathcal{H}^{\flat}{}_{\perp\overline{k}}$. In total, two propagating D.o.F remain as expected from Table 4.1

$$\begin{aligned}
2 = \frac{1}{2}\big(&80 - 2\times 10[\text{sPFC}] - 2\times(3+3)[\text{sSFC}] - 2\times(1+1+5)[\text{iPFC}] - (3+1+5)[\text{iPSC}] \\
&- 2\times(1+5)[\text{iSFC}] - (3+1+5)[\text{iSSC}]\big).
\end{aligned} \tag{4.61}$$

According to Table 4.1, these two D.o.F should be massless, and the power in the 2^+ sector of the propagator invites speculation as to whether they constitute a graviton. In Section 4.3.1 we were able to show that the one D.o.F of Case ^{*6}26 belonged unambiguously to the 0^- sector, but our method cannot be so straightforwardly applied to Case 17. This is ultimately related to the fact that the dependence of the PiC $\hat{\varphi}_{\overline{kl}}$ on the parallel torsion $\mathcal{T}^{i}_{\overline{kl}}$ in (4.8b), survives even when the defining constraints of Case 17 are imposed on the couplings. In the linear theory, this results in a conjugate SiC $\hat{\chi}^{\flat}{}_{\overline{kl}}$ which depends on the gradient of the torsion $\mathcal{D}^{\flat}{}_{[\overline{k}}\vec{\mathcal{T}}^{\flat}{}_{\overline{l}]}$. This is problematic when it comes to determining the linear multiplier $\hat{u}^{\flat}{}_{\overline{kl}}$ through the consistency condition (4.18). The result is a PDE in the multiplier whose inhomogeneous part results from a second-order Euler–Lagrange variation in the Poisson bracket, and likely involves gradients of the equal-time Dirac function. The remaining six determinable multipliers are not affected by this problem.

The ambiguity of $\hat{u}^{\flat}{}_{\overline{kl}}$ is problematic, as this multiplier lingers in the E.o.M. Even worse, the eight indeterminate multipliers $u^{\flat}$, ${}^{\mathrm{P}}u^{\flat}$ and $\tilde{u}^{\flat}{}_{\overline{kl}}$ associated with the iPFCs also feature prominently. In order to discover the J^P character of the propagating modes using Hamiltonian methods, one would have to fix the gauge.

We can draw some tentative conclusions just from the kinetic part of the Hamiltonian however. We see from Table 4.1 that the linear theory is unitary if only $\hat{\alpha}_5 < 0$. If unitarity is to be associated with the positive energy test, then the appearance of $\hat{\alpha}_5$ in (4.57) would suggest that $\hat{\alpha}_5 < 0$ serves to prevent the 1^- mode from becoming a ghost. By the same arguments, the 1^+ mode should be strongly coupled within the final shell of the linearised theory, since it would otherwise enter with negative kinetic energy. It is reassuring to see from Table 4.1 that the massless propagator does indeed have power in the 1^-, but not the 1^+ sectors. However, it also has power in the 2^+ sector, possibly inviting speculation that the theory may contain a spin-two graviton akin to that of Einstein. While $\tilde{\hat{\pi}}_{\overline{kl}}$ does feature in (4.57), it seems unlikely that this mode would independently propagate, since the unitarity of the theory does not depend on $\hat{\beta}_3$. We reiterate that these conclusions may ultimately depend on the gauge choice.

Finally, without a definite understanding of the propagating J^P, we are unable to say concretely whether fields will be activated or the PPM rank be field-dependent in the nonlinear theory. It is quite likely that these phenomena will occur, since we find in Appendix C.3 that the commutator of $\hat{\varphi}_{\overline{kl}}$ and ${}^{\mathrm{T}}\varphi_{\overline{klm}}$ depends on $\vec{\hat{\pi}}_{\overline{k}}$. This has precedent, since the 2^- commutator has spoiled all the theories in Section 4.3, but due to the lingering gauge ambiguity we denote it with $(\hat{\pi})$ rather than $(\hat{\pi}!)$ in (4.58).

4.4.2 Case 3

It should come as no surprise that Case 3 is a relaxation of Case 17, which admits an extra D.o.F. The kinetic part of the Hamiltonian is given by (4.57), in addition to the pseudoscalar term encountered in all the cases of Section 4.3. This is the usual massive 0^- mode, and comes with the no-ghost condition

$\hat{\alpha}_3 < 0$. The extra condition $\hat{\beta}_1$ will prevent this mode from being tachyonic. The nonlinear PPM is:

$$\left[\mathsf{M}^{(\text{Case 3})}_{\propto}\right] \approx \begin{array}{r|ccccc|l} & \varphi & \overset{\downarrow}{\hat{\varphi}_{\overline{kl}}} & \varphi_{\perp} & \tilde{\varphi}_{\perp\overline{kl}} & \overset{\downarrow}{{}^{\mathrm{T}}\varphi_{\overline{klm}}} & \\ \hline \varphi & \cdot & \hat{\pi} & \boxed{\cdot} & \cdot & \cdot & 1 \\ \rightarrow\ \hat{\varphi}_{\overline{kl}} & \hat{\pi} & \cdot & \hat{\pi} & \hat{\pi} & \hat{\pi}! & 3 \\ \varphi_{\perp} & \boxed{\cdot} & \hat{\pi} & \cdot & \cdot & \cdot & 1 \\ \tilde{\varphi}_{\perp\overline{kl}} & \cdot & \hat{\pi} & \cdot & \cdot & \cdot & 5 \\ \rightarrow\ {}^{\mathrm{T}}\varphi_{\overline{klm}} & \cdot & \hat{\pi}! & \cdot & \cdot & \cdot & 5 \\ \hline & 1 & 3 & 1 & 5 & 5 & \end{array} \tag{4.62}$$

It is clear that ${}^{\mathrm{P}}\varphi^{\flat}$ is no longer primarily constrained. The only change to the remaining SiCs of Case 17 is

$$\mathcal{H}^{\flat}{}_{\overline{kl}} \approx 2\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\overline{kl}} - \frac{1}{6}\epsilon^{\flat}{}_{\overline{klm}\perp}\eta^{\flat\overline{mn}}\mathcal{D}^{\flat}{}_{\overline{n}}{}^{\mathrm{P}}\hat{\pi}^{\flat} + \mathcal{D}^{\flat}{}_{[\overline{k}}\overset{\rightharpoonup}{\hat{\pi}}{}^{\flat}{}_{\overline{l}]}, \tag{4.63}$$

but $\mathcal{H}^{\flat}{}_{\alpha}$ is still satisfied. Overall, only the conjugate ${}^{\mathrm{P}}\varphi^{\flat}$ and ${}^{\mathrm{P}}\chi^{\flat}$ pair are removed, leaving three propagating D.o.F as expected from Table 4.1

$$\begin{aligned} 3 = \frac{1}{2}\big(&80 - 2\times 10[\text{sPFC}] - 2\times(3+3)[\text{sSFC}] - 2\times(1+1+5)[\text{iPFC}] - (3+5)[\text{iPSC}] \\ &- 2\times(1+5)[\text{iSFC}] - (3+5)[\text{iSSC}]\big). \end{aligned} \tag{4.64}$$

We draw the same conclusions from Case 3 as from Case 17 regarding the *vector* nature of the gravitational particle. This time however, we note the presence of ${}^{\mathrm{P}}\hat{\pi}$ in the nonlinear PPM, indicating that whatever the massless J^P, at least one gauge symmetry does not survive in the nonlinear regime.

4.5 Phenomenology

The results of Section 4.4 cast serious doubts on the health of even the massless theories considered here, on quite general grounds. We can in fact rule these theories out more conclusively on the basis of their cosmology. In general, this would be quite an arduous task, requiring a dedicated examination of all four E.o.M. However, we developed a mapping in Chapter 3 between the general quadratic torsion theory (3.2) and a torsion-free biscalar-tensor theory (the metrical analogue or MA), which immediately reveals the cosmological background. Beginning with the spatially flat FLRW line element (3.4), we align the unit timelike normal n^k to be perpendicular to the spatial slicing. Cosmological isotropy at the background level restricts only the 0^+ and 0^- torsion modes to propagate. From these modes respectively we now see that Eqs. (2.37) and (3.6) define the scalars $\phi \equiv \frac{2}{3}\mathcal{T}^{\overline{k}}{}_{\perp\overline{k}} - 2H$ and $\psi \equiv \frac{1}{6}\epsilon_{\overline{i}}{}^{\perp\overline{jk}}\mathcal{T}^{\overline{i}}{}_{\overline{jk}}$. These fields transform homogeneously and with the correct weight $\phi' = \Omega^{-1}\phi$, $\psi' = \Omega^{-1}\psi$ under changes of physical scale $b'^{i}{}_{\mu} = \Omega b^{i}{}_{\mu}$. In the usual second-order formulation of gravity on the curved V_4 spacetime $\mathcal{M}$, the MA is given by

$$\begin{aligned} L_{\mathrm{G}} \overset{\text{an}}{\equiv} &\left[\hat{\beta}_2 {m_{\mathrm{p}}}^2 + \frac{1}{4}(\hat{\alpha}_4 + \hat{\alpha}_6)\phi^2 - \frac{1}{4}(\hat{\alpha}_2 + \hat{\alpha}_3)\psi^2\right]R + 3(\hat{\alpha}_4 + \hat{\alpha}_6)X^{\phi\phi} - 3(\hat{\alpha}_2 + \hat{\alpha}_3)X^{\psi\psi} + \sqrt{|J_\mu J^\mu|} \\ &+ \frac{3}{4}(\hat{\alpha}_0 + 2\hat{\beta}_2){m_{\mathrm{p}}}^2\phi^2 - \frac{3}{4}(\hat{\alpha}_0 + 8\hat{\beta}_3){m_{\mathrm{p}}}^2\psi^2 + \frac{3}{8}(\hat{\alpha}_4 + \hat{\alpha}_6)\phi^4 + \frac{3}{8}(\hat{\alpha}_4 + \hat{\alpha}_6)\psi^4 \\ &- \frac{3}{4}\big((\hat{\alpha}_4 + \hat{\alpha}_6) + 2(\hat{\alpha}_2 + \hat{\alpha}_3)\big)\phi^2\psi^2, \end{aligned} \tag{4.65a}$$

$$J_\mu \equiv \big[(\hat{\alpha}_2 - \hat{\alpha}_3) - (\hat{\alpha}_4 - \hat{\alpha}_6)\big]\psi^3\nabla_\mu(\phi/\psi) - (\hat{\alpha}_0 + 2\hat{\beta}_2){m_{\mathrm{p}}}^2\nabla_\mu\phi, \tag{4.65b}$$

where we translate Eqs. (3.11a) and (3.11b) into the irreducible couplings using (B.24d).

We will restrict our attention to the massless theories. Following a reparameterisation to the weightless scalar $\zeta \equiv \sqrt{2}\phi/\psi$, we find the MA of Case 3 becomes

$$L_{\mathrm{G}} \stackrel{\mathrm{an}}{=} -3\hat{\alpha}_3 X^{\psi\psi} - \frac{1}{4}\hat{\alpha}_3\psi^2 R + 3\hat{\beta}_1 {m_{\mathrm{p}}}^2\psi^2 + \hat{\alpha}_3\psi^3\sqrt{|X^{\zeta\zeta}|} - \frac{3}{4}\hat{\alpha}_3\zeta^2\psi^4. \tag{4.66}$$

In this frame, we see that the MA can be partitioned into two. The first three terms in (4.66) describe a massive but *conformally coupled* scalar ψ. The fourth and fifth terms describe a *quadratic Cuscuton* ζ, which is conformally coupled by multiplication with the appropriate powers of ψ.

The quadratic *Cuscuton* is itself remarkable for replicating the cosmological background of the Einstein–Hilbert term [184]

$$c_1 {m_{\mathrm{p}}}^3\sqrt{|X^{\zeta\zeta}|} - c_2 {m_{\mathrm{p}}}^4\zeta^2 \stackrel{\mathrm{an}}{=} \frac{3{c_1}^2}{16c_2}{m_{\mathrm{p}}}^2 R, \tag{4.67}$$

This unlikely-looking relation may be verified by substituting the ζ-equation into the $g_{\mu\nu}$-equation on the LHS of (4.67), and comparing with the Friedmann equations that follow from the $g_{\mu\nu}$-equation on the RHS. We find that the bizarre characteristics of the *Cuscuton* can be taken further: when we replace the Planck mass with a dynamical scalar to obtain the conformally coupled quadratic *Cuscuton*, we replicate the cosmological background of the same scalar, conformally coupled to gravity

$$c_1\psi^3\sqrt{|X^{\zeta\zeta}|} - c_2\psi^4\zeta^2 \stackrel{\mathrm{an}}{=} \frac{9{c_1}^2}{4c_2}\Big(X^{\psi\psi} + \frac{1}{12}\psi^2 R\Big). \tag{4.68}$$

This result is very satisfying, but has fatal implications for the massless theories under consideration. By applying (4.68), we see that the fourth and fifth terms in (4.66) dynamically cancel with the first and second terms: the whole kinetic structure of the analogue theory vanishes! The same problem arises in Case 17, since the extra condition $\hat{\alpha}_3 = 0$ prevents the cancelling terms from appearing even at the level of (4.66). In both cases, the gravitational Lagrangian responsible for the cosmological background is a pure 0^- mass, and so the theories are *not viable*.

Notwithstanding the complete failure of the cases at hand, the result (4.68) suggests an interesting class of theories, of which Case 3 is a degenerate special case. From the general quadratic torsion theory (4.4) we impose

$$\hat{\alpha}_0 + 2\hat{\beta}_2 = \hat{\alpha}_4 + \hat{\alpha}_6 = 0, \tag{4.69}$$

noting from (4.10a) that the second constraint in (4.69) results in the single 0^+ PiC $\varphi_\perp \approx 0$. The cosmological analogue then becomes

$$\begin{aligned} L_{\mathrm{G}} \stackrel{\mathrm{an}}{=} & -\frac{1}{2}\hat{\alpha}_0 {m_{\mathrm{p}}}^2 R - 3(\hat{\alpha}_2 + \hat{\alpha}_3)\Big(X^{\psi\psi} + \frac{1}{12}\psi^2 R\Big) + \big[(\hat{\alpha}_2 - \hat{\alpha}_3) - (\hat{\alpha}_4 - \hat{\alpha}_6)\big]\psi^3\sqrt{|X^{\zeta\zeta}|} \\ & - \frac{3}{4}(\hat{\alpha}_2 + \hat{\alpha}_3)\zeta^2\psi^4 - \frac{3}{4}(\hat{\alpha}_0 + 8\hat{\beta}_3){m_{\mathrm{p}}}^2\psi^2. \end{aligned} \tag{4.70}$$

The interpretation of the first equality of (4.69) is now clear: it forces the Einstein–Hilbert term to appear equally both in the torsion theory and the cosmological background analogue[5]. We can then set $\hat{\alpha}_0 = 1$ to view these theories as additive modifications to the Einstein–Cartan or Einstein–Hilbert theories, respectively. In order to apply (4.68), we will strictly require that $\hat{\alpha}_2 + \hat{\alpha}_3 \neq 0$, i.e. that the

[5]Note that this is not *generally* guaranteed for general choices of the coupling constants, as discussed in Chapter 3.

0^- mode is not primarily constrained according to (4.10b). Under an appropriate rescaling of ψ to ξ, the cosmological background becomes the analogue of (4.1), i.e. Einstein's gravity conformally coupled to a scalar ξ, whose mass is

$$m_\xi^2 \equiv -\frac{(1+8\hat{\beta}_3)(\hat{\alpha}_2+\hat{\alpha}_3)}{8(\hat{\alpha}_3+\hat{\alpha}_4)(\hat{\alpha}_2+\hat{\alpha}_6)}{m_\text{p}}^2. \tag{4.71}$$

The theory (4.1) is of course widely studied in the context of inflation [231, 232]. In Einstein's theory, a non-minimal scalar coupling will tend to run, with the conformal value of 1/12 being a fixed point in the IR. This value is also used to preserve causality in a curved background, since it prevents a massive scalar from propagating along the light cone. We have shown that the cosmological background of the conformal scalar emerges as a consequence of the minimal constraints (4.69) on the quadratic torsion theory, where the scalar is interpreted as the 0^- part of the torsion, and the 0^+ part is primarily constrained.

We see also from (4.71) that the effect of the conformally coupled 0^- can be removed from the expansion history altogether. By setting $\hat{\alpha}_3+\hat{\alpha}_4=0$ or $\hat{\alpha}_2+\hat{\alpha}_6=0$, the mass m_ξ becomes infinite and one is left with the cosmological background of the pure Einstein gravity in (1). By inspecting Eqs. (4.10a) to (4.10f), we see that these choices can be imposed without primarily constraining the torsion modes in the general theory, including the 0^- mode. This raises the interesting question of whether torsion theories allow the cosmological background to be altered independently of the the perturbations. Note that Case 3 has just such a divergent mass, though the Einstein–Hilbert term never appears in the background analogue because of the universal constraint $\hat{\alpha}_0=0$ that appears to be required for PCR.

4.6 Closing remarks

In this chapter we have inspected the Hamiltonian structure of eight cases suggested from [152, 153] – as detailed in Table 4.1 – in both the linear and *nonlinear* regimes. Our principal findings may be summarised as follows;

1. All eight cases (and indeed all the cases proposed in [152, 153]) feature vanishing mass parameters. This greatly complicates the Hamiltonian analysis, compared to the 'minimal' cases previously treated in the literature.
2. The number of linear, propagating D.o.F are confirmed from [152, 153] for all eight cases.
3. With the exception of Case 17, all eight cases linearly propagate a massive pseudoscalar mode, and the unitarity conditions from [152, 153] correspond to the no-ghost and no-tachyon conditions on this mode.
4. The two massless modes propagated by Case 3 and Case 17 are identified with *vector*, rather than the hoped-for *tensor* modes.
5. With the possible exception of Case 20 and Case 32, all eight cases feature primary constraints which transition from first to second class when moving to the nonlinear regime. This signals at least a broken gauge symmetry, and possibly acausal behaviour and/or activation of any of the primarily unconstrained spin-parity sectors.
6. These primarily unconstrained spin-parity sectors include ghosts in all eight cases, according to the same conditions that ensure linearised unitarity.

7. Case 3 and Case 17 are not viable theories of gravity despite their massless modes, because they do not support a dynamical FLRW background.

These findings come with various caveats. Principally, while we implement the linearised Dirac–Bergmann algorithm to completion in all cases, we do not prosecute the nonlinear algorithm beyond the second set of links in the constraint chains. This level of analysis at least matches the earlier treatment of less complicated theories, in which all couplings are set to zero except those absolutely necessary to propagate whichever mode is under investigation [169]. Consequently, we cannot say for certain if the strongly coupled sectors and the ghost sectors coincide.

Separately, our definition of ghost sectors as set out in Appendix C.2 is based on the relevant quadratic momenta appearing as negative contributions to the Hamiltonian. We do not go so far as to quantise the theory and confirm that there are corresponding *physical* states which violate the unitarity of the S-matrix. Additional steps would presumably be required to draw completely safe conclusions, such as adding terms to fix the Poincaré gauge (and any other case-specific symmetries), and good ghosts to cancel the anomalies [219]. Meanwhile at the classical level, we mention that negative kinetic energy does not always imply instability.

We have also interpreted acausal behaviour, which is linked to the phenomenon of constraint bifurcation or field-dependent constraint structure [220], as a pathology. This need not always follow, as has been demonstrated for some special theories in recent decades [233]. For example, the characteristic surface of a degree of freedom is allowed to lie outside the light cone if it can be shown that the field does not carry information [183].

Even bearing these caveats in mind, the outlook for the remaining new torsion theories is not substantially improved by our results. Of the 58 novel theories in [152, 153], only 19 propagate the two massless D.o.F. Four of these additionally propagate a massive 0^- mode, while three instead propagate a massive 2^- mode. Of the remaining theories, 23 propagate only a massive 0^- mode. The selection in Table 4.1 thus appears reasonably representative of the linearised particle spectra. Since fundamental changes to the constraint structure are observed throughout most of the sample, we do not find new cause for optimism in the current study. The admission in Chapter 5 of primary constraints dependent on the Riemann–Cartan curvature will have mixed results. Certainly, such constraints will complicate the analysis. We have already seen in Section 4.4 that field-dependent primary constraints can invoke derivatives of the equal-time Dirac function. Ultimately, our findings so far are consistent with the predictions of Yo and Nester, who anticipate that generalising the quadratic torsion theory (3.2) beyond very minimal test cases (most of which also fail) serves only to protract the calculations [168, 169]. Even so, it might seem prudent to attempt to quantify the general chances of success given the results of this chapter alone: we provide a heuristic discussion along these lines in Appendix C.4.

The tentative vector nature of the massless modes in Case 3 and Case 17 is potentially problematic. We recall that Poincaré invariance prohibits a matter amplitude involving soft gravitons of spin $J > 2$, while $J = 0$ gravitons are ruled out by matter coupling [234]. Odd J are supposed to give rise to *repulsive* long-range forces, leading to the expectation of a tensor graviton [67]. Plausibly, the J^P character will be gauge dependent, but it is difficult to see how this might change the sign of the Green's function. We will see in Chapter 5 that this troubling feature is not generic to the remaining massless cases.

Finally, we observed that the theories with massless modes could be written off instantly using the scalar-tensor analogue theory which replicates the background cosmology. As a by-product, our analysis suggested an interesting new class of quadratic torsion theories which mimic the background of the conformal inflaton, though not motivated by unitarity or renormalisability. It must be emphasised that the catastrophic failure of Case 3 and Case 17 is *not* common to the remaining theories in [152, 153]. We mention in particular Case 2, which propagates two massless modes and the massive pseudoscalar, and Case 16, a special case in which the pseudoscalar is non dynamical. These theories form a complementary pair to Case 3 and Case 17 in many respects, but they have an *excellent* cosmological background as shown in Chapters 2 and 3. These cases call for a more dedicated Hamiltonian analysis, which we perform next in Chapter 5.

Chapter 5

Gauge theory with geometric multipliers

Abridged from W. E. V. Barker, A. N. Lasenby, M. P. Hobson and W. J. Handley, *in prep.* for submission to Physical Review D.

5.1 Introduction

The findings presented in Chapter 4 highlight dynamical challenges associated with $\mathrm{PGT}^{\mathrm{q},+}$ in general, and the loss of the Einstein–Hilbert term in particular. We will not be surprised to learn that the supposedly viable theories developed throughout Chapters 2 and 3 – which differ from those considered in Chapter 4 principally in the complexity of their constraints – are not immune to these problems. In this final chapter therefore we seek an extension of the $\mathrm{PGT}^{\mathrm{q},+}$ which tends to ameliorate the nonlinear proliferation of propagating D.o.F.

We will first require this extension to be *minimal.* Many attractive options present, for example, when one considers alternatives to the Poincaré gauge group. However, since these alternatives are chiefly realised within existing frameworks such as WGT [194, 195], eWGT [92] and MAGT [235], they do not lie comfortably within our scope. Nor shall we augment the $\mathrm{PGT}^{\mathrm{q},+}$ with new dynamical fields, such as the scalar added by Horndeski to Einstein's theory [187]. That technique is better suited to effective rather than fundamental theories, and indeed we exploited it as such in Chapter 3. In fact a particularly conservative approach is suggesed already within the same chapter, in the form of *teleparallel gravity* [236]. We recall from Section 3.3.1 that the teleparallel form of GR has total Lagrangian[1]

$$L_{\mathrm{T}} = \frac{1}{2}{m_{\mathrm{p}}}^2\mathbb{T} + {\lambda_{ij}}^{kl}{\mathcal{R}^{ij}}_{kl} + L_{\mathrm{M}}, \quad \mathbb{T} \equiv \frac{1}{4}\mathcal{T}_{ijk}\mathcal{T}^{ijk} + \frac{1}{2}\mathcal{T}_{ijk}\mathcal{T}^{jik} - \mathcal{T}_i\mathcal{T}^i. \tag{5.1}$$

The dynamical part $\mathbb{T}$ of (5.1) is purely quadratic in torsion. There is however added to this Lagrangian a kinematic term, which suppresses the whole Riemann–Cartan curvature by means of 36 multiplier fields ${\lambda_{ij}}^{kl}$. In the geometric interpretation, the multipliers constrain the rich Riemann–Cartan geometry U_4

[1] Note that our conventions for multipliers, which we take to be tensors, will differ from those used in [67], where they are treated as densities. We believe the resulting dynamics to be unchanged.

to that of Weitzenböck T_4, eliminating unwanted modes in the process. They do not however appear as propagating D.o.F in the final counting: those that persist in the E.o.M do so as *determined* quantities, so that those same equations are devoid of physical content while the rest propagate nothing more than the 2^+ graviton. In the grand picture, the theory (5.1) is rightly considered a PGT, since the multipliers play a *restrictive* rôle. It is on geometry-constraining multipliers therefore, that we will focus.

We also require our extension to be *general*. The motivating quantum mechanical properties obtained in [152] are very sensitive to the Lagrangian structure. It would seem unlikely therefore that the nonlinear dynamics of our favoured theories can be pacified while preserving their unitarity and renormalisability at linear order. A general theory of geometry-constraining multipliers, or *geometric multipliers*, will instead provide a solid foundation for a fresh propagator analysis of $\text{PGT}^{\text{q},+}$ in the future.

Even so, it still seems natural to try applying the multipliers to the leading theories of Chapters 2 and 3, and this will turn out to be rewarding for unexpected reasons. We briefly now take stock of these theories in the language of Chapter 4, with the aid of Table 2.1 and (B.23i). Recall from Fig. 2.1 that the linearised Case 2 was the most general theory with favourable quantum mechanics: the nonlinear theory was then shown to feature both dark energy and dark radiation in an exactly Einsteinian cosmology. Case 2 is reached as a restriction of (4.4) by the constraints

$$\hat{\alpha}_1 + 2\hat{\alpha}_6 = \hat{\alpha}_2 - 2\hat{\alpha}_6 = \hat{\alpha}_4 + \hat{\alpha}_6 = \hat{\beta}_1 = 0. \tag{5.2}$$

Using (B.24d) we obtain from Chapter 2 and Chapter 3 the further conditions which shore up the linearised unitarity and nonlinear phenomenology

$$\hat{\alpha}_3 = 0, \quad \hat{\alpha}_6 < 0, \quad \hat{\beta}_3 > 0, \quad \Lambda = -\frac{2\hat{\beta}_3}{\hat{\alpha}_6} {m_{\text{p}}}^2, \quad \hat{\beta}_2 = -\frac{2}{3}, \tag{5.3}$$

and together with Eq. (5.2) these constitute Class ${}^{2}\text{A}^*$.

Let us illustrate the potential problems with our preferred theories by focusing instead on Case 16 – reached by further setting $\hat{\beta}_3 = 0$ in (4.4) – since the more restricted theory makes for a simpler canonical analysis. Recall that this theory is the basis for Class ${}^{3}\text{C}^*$, which we see from (5.3) differs from Class ${}^{2}\text{A}^*$ by lacking emergent dark energy. We identify the following PiCs

$$\overset{\wedge}{\varphi}_{\overline{ij}} = \frac{1}{J}\overset{\wedge}{\hat{\pi}}_{\overline{ij}} \approx \overset{\sim}{\varphi}_{\overline{ij}} = \frac{1}{J}\overset{\sim}{\hat{\pi}}_{\overline{ij}} \approx \varphi_{\perp} = \frac{1}{J}\hat{\pi}_{\perp} - 4\hat{\alpha}_6\underline{\mathcal{R}} \approx {}^{\text{T}}\varphi_{\overline{ijk}} = \frac{1}{J}{}^{\text{T}}\hat{\pi}_{\overline{ijk}} + 16\hat{\alpha}_6{}^{\text{T}}\mathcal{R}_{\perp\overline{ijk}} \approx 0, \tag{5.4}$$

which leave the following quadratic momenta terms in the super-Hamiltonian

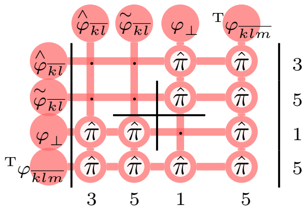
$$\mathcal{H}_{\text{T}} = \frac{b}{96}\left(\frac{6\overset{\rightharpoonup}{\varphi}_{\overline{k}}\overset{\rightharpoonup}{\varphi}^{\overline{k}}}{\hat{\alpha}_5 - \hat{\alpha}_6} + \frac{12\overset{\wedge}{\varphi}_{\perp\overline{kl}}\overset{\wedge}{\varphi}^{\perp\overline{kl}}}{\hat{\alpha}_5 + 2\hat{\alpha}_6} - \frac{4\overset{\sim}{\varphi}_{\perp\overline{kl}}\overset{\sim}{\varphi}^{\perp\overline{kl}}}{\hat{\alpha}_6} - \frac{{}^{\text{P}}\varphi^2}{\hat{\alpha}_3 + 2\hat{\alpha}_6} + \frac{4\big(\varphi^2 + 9\varphi_{\perp\overline{k}}\varphi^{\perp\overline{k}}\big)}{\hat{\beta}_2}\right) + \text{fields}, \tag{5.5}$$

while in the absence of mass parameters we find only one (no-ghost) unitarity condition

$$\hat{\alpha}_6(\hat{\alpha}_5 + 2\hat{\alpha}_6)(\hat{\alpha}_5 - \hat{\alpha}_6) < 0. \tag{5.6}$$

The condition (5.6) appears to refer to the first three square momenta in (5.5), *including* the momentum of the 2^+ tordion mode. This could be cause for optimism, since in Section 4.4 we were disappointed to find that the unitarity condition protected only the unwanted 1^- mode of Case 17. However the troubles of Chapter 4 resurface in the nonlinear PPM, which is written[2] on the PiC shell:

$$\left[\mathsf{M}^{(\text{Case 16})}_{\propto}\right] \approx \tag{5.7}$$



Thus, the PiCs (5.4) all fail to commute at the nonlinear level, with the entries in (5.7) schematically representing linear combinations of the momenta and field strengths, which we give in Appendix C.3. There are other challenges which hinder the remaining steps in the analysis of Case 16, and these are set out in Appendix C.7; the most obvious objection however would seem to be the *purely nonlinear population* of (5.7). We note without showing it explicitly that this situation is not improved in Case 2, or by imposing the phenomenological constraints of (5.3).

Despite appearances, whether the nonlinear fragility of the linear analyses poses a genuine problem is actually far *less* clear in these cases than it was in Chapter 4. We discovered in Chapters 2 and 3 that our theory functions best in a dynamically emergent state of constant axial torsion. This *correspondence solution* (CS) is not the same vacuum as that considered in [152, 153]: its stable existence is both a highly desirable *and* inherently nonlinear phenomenon. Until the linear QFT near the CS is investigated, we cannot claim on the grounds of (5.7) that the nonlinear Hamiltonian structure is actually sick. If anything, our current knowledge of the CS vacuum should give us the freedom to entertain modifications to the theory which actually damage the linear unitarity and PCR properties. Accordingly, we still find merit in applying the multipliers to Case 2 and Case 16 because (i) the general Hamiltonian effects of the multipliers are illustrated and (ii) the resulting 'bypass' theory in (4) will receive a specific phenomenological benefit in the form of a quick-to-obtain Newtonian limit.

The remainder of this chapter is set out as follows. In Sections 5.2.1 to 5.2.2 we set out the general theory of geometric multipliers in the Lagrangian formulation. In Sections 5.2.4 to 5.2.5 we apply the technique to the favoured theories, phenomenologically excluding most of the multipliers and setting the weak gravity limit. We dissect the new, general Hamiltonian structure in Sections 5.3.2 to 5.3.3, showing the mechanism by which the multipliers generally soften the dynamical transition from linear to nonlinear gravity. In Section 5.3.4 we perform the canonical analysis of the bypass theory, and demonstrate this mechanism in action. Conclusions follow in Section 5.4.

5.2 The Lagrangian picture

5.2.1 Developing the formalism

The central innovation of the present chapter is the covariant restriction of the Riemann–Cartan geometry through the introduction of *geometric multipliers*. An additional 60 gravitational D.o.F are added to the PGT via the multiplier fields $\lambda^{i}{}_{jk}$ and $\lambda^{ij}{}_{kl}$, which share the symmetries and dimensions of the

[2] The colours indicate that none of the J^P PiC sectors appear safe at nonlinear order: they are introduced for later comparison with (5.73).

Riemann–Cartan and torsion tensors. The new gravitational Lagrangian is written as

$$
\begin{aligned}
L_{\mathrm{G}} = &\sum_{I=1}^{6}\left(\hat{\alpha}_I \mathcal{R}^{ij}{}_{kl} + \bar{\alpha}_I \lambda^{ij}{}_{kl}\right){}^{I}\hat{\mathcal{P}}_{ij}{}^{kl}{}_{nm}{}^{pq}\mathcal{R}^{nm}{}_{pq} \\
&+ {m_{\mathrm{p}}}^2 \sum_{M=1}^{3}\left(\hat{\beta}_M \mathcal{T}^{i}{}_{jk} + \bar{\beta}_M \lambda^{i}{}_{jk}\right){}^{M}\hat{\mathcal{P}}_{i}{}^{jk}{}_{l}{}^{nm}\mathcal{T}^{l}{}_{nm}\,,
\end{aligned}
\tag{5.8}
$$

where any nonvanishing $\{\bar{\alpha}_I\}$ and $\{\bar{\beta}_M\}$ switch off the various irreducible representations of $\mathrm{SO}^+(1,3)$ which are contained within the field strengths. The $2^3 \times 2^6$ configurations of these boolean 'switches' allow the greatest possible control over theory beyond the nine 'dials' which define the original $\mathrm{PGT}^{\mathrm{q},+}$, whilst maintaining general covariance.

In order to efficiently and thoroughly discuss the new general theory (5.8), we must create a more formal notation than that previously used in Chapter 4. We recall that the indices I, J, K and L label the $\mathrm{SO}^+(1,3)$ irreps of $\mathcal{R}^{ij}{}_{kl}$, ranging from one to six, and we allocate M, N, O and P to label those of $\mathcal{T}^{i}{}_{jk}$, ranging from one to three. We now also introduce A, B, C and D to span the SO(3) irreps in the *rotational context*, such as those contained within $\hat{\pi}_{ij}{}^{\overline{k}}$, $\mathcal{R}^{ij}{}_{\overline{kl}}$ and $\mathcal{R}^{ij}{}_{\perp\overline{l}}$, and which are 0^+, 0^-, 1^+, 1^-, 2^+ and 2^-. We will use E, F, G and H to span these *same* irreps in the *translational context*, i.e. wherever such irreps are present in $\hat{\pi}_{i}{}^{\overline{k}}$, $\mathcal{T}^{i}{}_{\overline{kl}}$ and $\mathcal{T}^{i}{}_{\perp\overline{l}}$. Care must be taken, since various of the six spin-parity irreps are missing from various objects in the translational sector, and summations over the new indices are assumed to take this into account implicitly. Using this notation, we next introduce the 'human readable' projections as denoted with a háček ${}^{A}\hat{\pi}_{\acute{l}} \equiv {}^{A}\check{\mathcal{P}}_{\acute{l}}{}^{ij}{}_{\overline{k}}\hat{\pi}_{ij}{}^{\overline{k}}$, ${}^{E}\hat{\pi}_{\acute{l}} \equiv {}^{E}\check{\mathcal{P}}_{\acute{l}}{}^{i}{}_{\overline{k}}\hat{\pi}_{i}{}^{\overline{k}}$, etc. These obtain the convenient $\hat{\pi}_{\perp}$, $\overset{\wedge}{\hat{\pi}}_{\perp\overline{kl}}$, $\overset{\sim}{\hat{\pi}}_{\perp\overline{kl}}$ etc., where a *variable* number of indices is denoted by $\acute{u}$, $\acute{v}$, $\acute{w}$, etc., as with Lin's notation in [152]. To account for missing irreps, we define placeholder projections within the translational sector

$$
{}^{0^+}\check{\mathcal{P}}_{\acute{v}}{}^{i}{}_{\overline{jk}} \equiv {}^{2^+}\check{\mathcal{P}}_{\acute{v}}{}^{i}{}_{\overline{jk}} \equiv {}^{0^-}\check{\mathcal{P}}_{\acute{v}}{}^{i}{}_{\overline{k}} \equiv {}^{2^-}\check{\mathcal{P}}_{\acute{v}}{}^{i}{}_{\overline{k}} \equiv 0. \tag{5.9}
$$

There is a related *complete* set of operators as denoted with a circumflex. It is very convenient to describe relations between both sets of operators using the dimensionless numbers $\{c^{\parallel}_{A}\}$, $\{c^{\parallel}_{E}\}$, $\{c^{\perp}_{A}\}$, $\{c^{\perp}_{E}\}$, which are close to unity

$$
\begin{aligned}
{}^{A}\hat{\mathcal{P}}_{ij}{}^{\overline{kp}}{}_{lm}{}^{\overline{nq}} &\equiv c^{\parallel}_{A}\,{}^{A}\check{\mathcal{P}}^{\acute{u}}{}_{ij}{}^{\overline{kp}}\,{}^{A}\check{\mathcal{P}}_{\acute{u}lm}{}^{\overline{nq}}, &
{}^{A}\hat{\mathcal{P}}_{ij}{}^{\overline{k}}{}_{lm}{}^{\overline{n}} &\equiv c^{\perp}_{A}\,{}^{A}\check{\mathcal{P}}^{\acute{u}}{}_{ij}{}^{\overline{k}}\,{}^{A}\check{\mathcal{P}}_{\acute{u}lm}{}^{\overline{n}}, \\
c^{\parallel}_{A}\delta^{\acute{v}}_{\acute{u}} &\equiv {}^{A}\check{\mathcal{P}}_{\acute{u}ij}{}^{\overline{kp}}\,{}^{A}\check{\mathcal{P}}^{\acute{v}ij}{}_{\overline{kp}}, &
c^{\perp}_{A}\delta^{\acute{v}}_{\acute{u}} &\equiv {}^{A}\check{\mathcal{P}}_{\acute{u}ij}{}^{\overline{p}}\,{}^{A}\check{\mathcal{P}}^{\acute{v}ij}{}_{\overline{p}}, \\
{}^{E}\hat{\mathcal{P}}_{i}{}^{\overline{kp}}{}_{l}{}^{\overline{nq}} &\equiv c^{\parallel}_{E}\,{}^{E}\check{\mathcal{P}}^{\acute{u}}{}_{i}{}^{\overline{kp}}\,{}^{E}\check{\mathcal{P}}_{\acute{u}l}{}^{\overline{nq}}, &
{}^{E}\hat{\mathcal{P}}_{i}{}^{\overline{k}}{}_{l}{}^{\overline{n}} &\equiv c^{\perp}_{E}\,{}^{E}\check{\mathcal{P}}^{\acute{u}}{}_{i}{}^{\overline{k}}\,{}^{E}\check{\mathcal{P}}_{\acute{u}l}{}^{\overline{n}}, \\
c^{\parallel}_{E}\delta^{\acute{v}}_{\acute{u}} &\equiv {}^{E}\check{\mathcal{P}}_{\acute{u}i}{}^{\overline{kp}}\,{}^{E}\check{\mathcal{P}}^{\acute{v}i}{}_{\overline{kp}}, &
c^{\perp}_{E}\delta^{\acute{v}}_{\acute{u}} &\equiv {}^{E}\check{\mathcal{P}}_{\acute{u}i}{}^{\overline{p}}\,{}^{E}\check{\mathcal{P}}^{\acute{v}i}{}_{\overline{p}}.
\end{aligned}
\tag{5.10}
$$

These complete operators are more cumbersome in their actual form, but very useful for formal calculations. Most importantly, we introduce a compact notation for the linear combinations of coupling constants which arise frequently at all levels of analysis. There are eight matrices, again populated by numbers close to unity

$$
\begin{aligned}
{}^{A}\check{\mathcal{P}}_{\acute{p}}{}^{lm}{}_{\overline{nq}}\,{}^{I}\hat{\mathcal{P}}_{lm}{}^{\overline{nq}}{}_{ij}{}^{\overline{rk}} &\equiv M^{\|\|}_{AI}\,{}^{A}\check{\mathcal{P}}_{\acute{p}ij}{}^{\overline{rk}}, &\quad {}^{A}\check{\mathcal{P}}_{\acute{p}}{}^{lm}{}_{\overline{n}}\,{}^{I}\hat{\mathcal{P}}_{lm}{}^{\perp\overline{n}}{}_{ij}{}^{\overline{rk}} &\equiv M^{\perp\|}_{AI}\,{}^{A}\check{\mathcal{P}}_{\acute{p}ij}{}^{\overline{rk}},\\
{}^{A}\check{\mathcal{P}}_{\acute{p}}{}^{lm}{}_{\overline{nq}}\,{}^{I}\hat{\mathcal{P}}_{lm}{}^{\overline{nq}}{}_{ij}{}^{\perp\overline{k}} &\equiv M^{\|\perp}_{AI}\,{}^{A}\check{\mathcal{P}}_{\acute{p}ij}{}^{\overline{k}}, &\quad {}^{A}\check{\mathcal{P}}_{\acute{p}}{}^{lm}{}_{\overline{n}}\,{}^{I}\hat{\mathcal{P}}_{lm}{}^{\perp\overline{n}}{}_{ij}{}^{\perp\overline{k}} &\equiv M^{\perp\perp}_{AI}\,{}^{A}\check{\mathcal{P}}_{\acute{p}ij}{}^{\overline{k}},\\
{}^{E}\check{\mathcal{P}}_{\acute{p}}{}^{l}{}_{\overline{nq}}\,{}^{M}\hat{\mathcal{P}}_{l}{}^{\overline{nq}}{}_{i}{}^{\overline{rk}} &\equiv M^{\|\|}_{EM}\,{}^{E}\check{\mathcal{P}}_{\acute{p}i}{}^{\overline{rk}}, &\quad {}^{E}\check{\mathcal{P}}_{\acute{p}}{}^{l}{}_{\overline{q}}\,{}^{M}\hat{\mathcal{P}}_{l}{}^{\perp\overline{q}}{}_{i}{}^{\overline{rk}} &\equiv M^{\perp\|}_{EM}\,{}^{E}\check{\mathcal{P}}_{\acute{p}i}{}^{\overline{rk}},\\
{}^{E}\check{\mathcal{P}}_{\acute{p}}{}^{l}{}_{\overline{nq}}\,{}^{M}\hat{\mathcal{P}}_{l}{}^{\overline{nq}}{}_{i}{}^{\perp\overline{k}} &\equiv M^{\|\perp}_{EM}\,{}^{E}\check{\mathcal{P}}_{\acute{p}i}{}^{\perp\overline{k}}, &\quad {}^{E}\check{\mathcal{P}}_{\acute{p}}{}^{l}{}_{\overline{q}}\,{}^{M}\hat{\mathcal{P}}_{l}{}^{\perp\overline{q}}{}_{i}{}^{\perp\overline{k}} &\equiv M^{\perp\perp}_{EM}\,{}^{E}\check{\mathcal{P}}_{\acute{p}i}{}^{\perp\overline{k}},
\end{aligned}
\tag{5.11}
$$

which encode the *transfer* of SO(3) projections *through* the $\mathrm{SO}^{+}(1,3)$ projections. With these matrices we obtain various *transfer couplings*, using the obvious notation $\hat{\alpha}^{\|\|}_{A} \equiv \sum_I M^{\|\|}_{AI}\hat{\alpha}_I$, $\bar{\beta}^{\perp\|}_{E} \equiv \sum_M M^{\perp\|}_{EM}\bar{\beta}_M$, etc. Due to (5.9), the relations (5.11) do not fully define these quantities and we again supplement with the *vanishing* placeholder couplings $\bar{\beta}^{\perp\|}_{0^+}$, $\bar{\beta}^{\perp\|}_{2^+}$, $\bar{\beta}^{\|\perp}_{0^-}$, $\bar{\beta}^{\|\perp}_{2^-}$, $\hat{\beta}^{\perp\|}_{0^+}$, $\hat{\beta}^{\perp\|}_{2^+}$, $\hat{\beta}^{\|\perp}_{0^-}$ and $\hat{\beta}^{\|\perp}_{2^-}$. Explicit formulae for all transfer couplings are provided in Appendix B.8. We shall show in Section 5.3 that the canoncial structure of $\mathrm{PGT}^{\mathrm{q},+}$ and the geometric multiplier extension in (5.8) can be fully understood through the transfer couplings and their relations Eqs. (B.27) and (B.28). Finally, we will add two more items of formalism by defining the functions

$$
\mu(x) \equiv \begin{cases} x^{-1}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0, \end{cases} \qquad \nu(x) \equiv 1 - |\mathrm{sgn}(\mu(x))|. \tag{5.12}
$$

These functions allow for a general discussion of constrained quantities in the Hamiltonian picture, and in particular the function $\mu(x)$ is not new, being defined already by Blagojević and Nikolić in [226], as part of the *if-constraint formalism.*

5.2.2 The gravitational field equations

We will begin our discussion of the physical structure of the theory (5.8) by considering the Lagrangian field equations. We borrow from [67] the definition of the generalised momenta

$$
\pi_i{}^{kl} \equiv \frac{\partial bL_{\mathrm{G}}}{\partial\partial_\nu b^i{}_\mu} b^k{}_\mu b^l{}_\nu \equiv -\frac{\partial bL_{\mathrm{G}}}{\partial T^i{}_{\mu\nu}} b^k{}_\mu b^l{}_\nu \equiv -2m_{\mathrm{p}}{}^2 b \sum_M \left(2\hat{\beta}_M \mathcal{T}^j{}_{nm} + \bar{\beta}_M \lambda^j{}_{nm}\right) {}^{M}\hat{\mathcal{P}}_j{}^{nm}{}_i{}^{kl}, \tag{5.13a}
$$

$$
\pi_{ij}{}^{kl} \equiv \frac{\partial bL_{\mathrm{G}}}{\partial\partial_\nu A^{ij}{}_\mu} b^k{}_\mu b^l{}_\nu \equiv -\frac{\partial bL_{\mathrm{G}}}{\partial R^{ij}{}_{\mu\nu}} b^k{}_\mu b^l{}_\nu \equiv -4b \sum_I \left(2\hat{\alpha}_I \mathcal{R}^{pq}{}_{nm} + \bar{\alpha}_I \lambda^{pq}{}_{nm}\right) {}^{I}\hat{\mathcal{P}}_{pq}{}^{nm}{}_{ij}{}^{kl}, \tag{5.13b}
$$

and we note for later convenience that, for *any* values adopted by the various couplings, these quantities can be shown after a somewhat lengthy calculation to satisfy the identities

$$
\mathcal{T}_{[j|pq}\,\pi_{|i]}{}^{pq} - 2\mathcal{T}^p{}_{k[i|}\,\pi_p{}^k{}_{|j]} \equiv \mathcal{R}^k{}_{[i|pq}\pi_{k|j]}{}^{pq} + \mathcal{R}^{pq}{}_{k[i|}\pi_{pq}{}^k{}_{|j]} \equiv 0. \tag{5.14}
$$

In terms of the generalised momenta we then obtain the stress-energy and spin field equations of the theory in the presence of matter sources

$$
\tau^\mu{}_\nu \equiv h_k{}^\mu \frac{\delta bL_{\mathrm{M}}}{\delta h_k{}^\nu} \equiv -\frac{\delta bL_{\mathrm{M}}}{\delta b^k{}_\mu} b^k{}_\nu, \qquad \tau^\nu{}_i = -D_\mu \pi_i{}^{\nu\mu} + \mathcal{T}^p_{ki}\,\pi_p{}^{k\nu} + \frac{1}{2}\mathcal{R}^{pq}{}_{ki}\pi_{pq}{}^{k\nu} + bL_{\mathrm{G}} h_i{}^\nu, \tag{5.15a}
$$

$$
\sigma^\mu{}_{ij} \equiv -\frac{\delta bL_{\mathrm{M}}}{\delta A^{ij}{}_\mu}, \qquad \sigma^\nu{}_{ij} = -D_\mu \pi_{ij}{}^{\nu\mu} + 2\pi_{[ij]}{}^\nu. \tag{5.15b}
$$

These equations may be manipulated further. We see that the divergence of the spin equation (5.15b) is

$$D_\mu \sigma^\mu{}_{ij} = 2D_\mu \pi_{[ij]}{}^\mu + \mathcal{R}^k{}_{[i|pq}\pi_{k|j]}{}^{pq}, \tag{5.16}$$

and we can expand the energy-momentum equation (5.15a) to give

$$\tau^j{}_i = -D_\mu \pi_i{}^{j\mu} - \frac{1}{2}\mathcal{T}^j{}_{pq}\,\pi_i{}^{pq} + \mathcal{T}^p{}_{ki}\,\pi_p{}^{kj} + \frac{1}{2}\mathcal{R}^{pq}{}_{ki}\pi_{pq}{}^{kj} + bL_{\mathrm{G}}\delta^j_i. \tag{5.17}$$

However, by considering the skew-symmetric part of (5.17) and the conservation law $D_\mu \sigma^\mu{}_{ij} \equiv 2\tau_{[ij]}$, we see that there is another relation

$$D_\mu \sigma^\mu{}_{ij} = 2D_\mu \pi_{[ij]}{}^\mu - \mathcal{R}^{pq}{}_{k[i|}\pi_{pq}{}^k{}_{|j]}. \tag{5.18}$$

From (5.18) and (5.16) we can use the identities (5.14) to confirm the gravitational equivalent of the conservation law, i.e. that six of the field equations are in fact shared between (5.15a) and (5.15b). In the simple case of the teleparallel theory, we note that this result may be used to identify the so-called 'λ symmetry', i.e. the parts of $\lambda_i{}^{jk}$ which remain dynamically undetermined [236].

The most salient consequence of the geometric multipliers in (5.8) follows from their own field equations, which suppress various parts of the Riemann–Cartan and torsion tensors. The first opportunity to employ the transfer couplings from Section 5.2.1 arises when we decompose these field equations into their respective SO(3) irreps, to give

$$\begin{pmatrix} \bar{\alpha}^{\parallel\parallel}_A & \bar{\alpha}^{\parallel\perp}_A \\ \bar{\alpha}^{\perp\parallel}_A & \bar{\alpha}^{\perp\perp}_A \end{pmatrix} \begin{pmatrix} {}^A\check{\mathcal{P}}_{\acute{\imath}nm}{}^{\overline{pq}}\mathcal{R}^{nm}{}_{\overline{pq}} \\ 2\,{}^A\check{\mathcal{P}}_{\acute{\imath}nm}{}^{\bar{q}}\mathcal{R}^{nm}{}_{\perp\bar{q}} \end{pmatrix} \approx \begin{pmatrix} \bar{\beta}^{\parallel\parallel}_E & \bar{\beta}^{\parallel\perp}_E \\ \bar{\beta}^{\perp\parallel}_E & \bar{\beta}^{\perp\perp}_E \end{pmatrix} \begin{pmatrix} {}^E\check{\mathcal{P}}_{\acute{\imath}n}{}^{\overline{pq}}\mathcal{T}^n{}_{\overline{pq}} \\ 2\,{}^E\check{\mathcal{P}}_{\acute{\imath}n}{}^{\bar{q}}\mathcal{T}^n{}_{\perp\bar{q}} \end{pmatrix} \approx \mathbf{0}. \tag{5.19}$$

The consequences of the geometric multipliers are thus fully encoded by the pre-multiplying matrices in (5.19). Less formally, we provide in Appendix B.8 a translation of (5.19) in terms of the 'human readable' SO(3) representations of the Riemann–Cartan curvature and torsion.

Multipliers imposed to correct pathologies in the original $\mathrm{PGT}^{\mathrm{q},+}$ should not intefere with the desirable phenomenology, as established for example in Chapters 2 and 3. This principle of *selective non-interference* can be implemented by choosing the multiplier couplings so that

$$\sum_I \bar{\alpha}_I{}^I\hat{\mathcal{P}}_{ij}{}^{kl}{}_{pq}{}^{mn}\mathcal{R}^{pq}{}_{nm} \approx \sum_M \bar{\beta}_M{}^M\hat{\mathcal{P}}_i{}^{kl}{}_p{}^{mn}\mathcal{T}^p{}_{nm} \approx 0, \tag{5.20}$$

on the shell defined by all desirable solutions to the original theory. These solutions are then *still valid* for all multiplier extensions of the original theory which obey (5.20), so long as the multipliers themselves solve the coupled, homogeneous, first-order linear system

$$-D_\mu \Lambda_i{}^{\nu\mu} + \mathcal{T}^p{}_{ki}\,\Lambda_p{}^{k\nu} + \mathcal{R}^{pq}{}_{ki}\Lambda_{pq}{}^{k\nu} \approx -D_\mu \Lambda_{ij}{}^{\nu\mu} + \Lambda_{[ij]}{}^\nu \approx 0, \tag{5.21}$$

which is derived from Eqs. (5.15a) and (5.15b), and expressed in terms of the 'employed' multiplier D.o.F

$$\Lambda_{ij}{}^{kl} \equiv \sum_I \bar{\alpha}_I{}^I\hat{\mathcal{P}}_{ij}{}^{kl}{}_{pq}{}^{mn}\lambda^{pq}{}_{nm}, \quad \Lambda_i{}^{kl} \equiv m_{\mathrm{p}}{}^2\sum_M \bar{\beta}_M{}^M\hat{\mathcal{P}}_i{}^{kl}{}_p{}^{mn}\lambda^p{}_{nm}. \tag{5.22}$$

Formally, the system (5.21) can always be satisfied (e.g. with vanishing multipliers), though attention must still be paid to the uniqueness of such solutions for a given spacetime symmetry, along with the physical interpretation of the multipliers.

5.2.3 Exact solutions to the novel theories

We are now ready to return to the theories developed in Chapters 2 and 3. Before exploring the multiplier extensions to these theories it is appropriate to summarise their exact solutions. By identifying solutions which are thought to be phenomenologically mandatory, we can implement the principle of selective non-interference and so draw up a shortlist of viable multiplier configurations.

We will first introduce novel solutions to Case 16 which describe null pp-waves on the Minkowski background [182]. While falling somewhat outside the scope of our earlier cosmological analyses, these waves survive the resulting parameter constraints listed in (5.3). The wave solutions are formulated in the *Brinkmann* gauge, rather than the more popular transverse-traceless (TT) setup [237]. In a Cartesian coordinate system, the rotation gauge is chosen so that the local Lorentz and coordinate bases are aligned, and we define 'perpendicular' and 'null' vectors as

$$\boldsymbol{e}_t \equiv \hat{\boldsymbol{e}}_0, \quad \boldsymbol{e}_x \equiv \hat{\boldsymbol{e}}_1, \quad \boldsymbol{e}_y \equiv \hat{\boldsymbol{e}}_2, \quad \boldsymbol{e}_z \equiv \hat{\boldsymbol{e}}_3, \quad \boldsymbol{e}_\perp \equiv \cos(\theta)\boldsymbol{e}_x + \sin(\theta)\boldsymbol{e}_y, \quad \boldsymbol{e}_+ \equiv \boldsymbol{e}_t + \boldsymbol{e}_z. \tag{5.23}$$

We will denote the 'wave coordinate' as $\tau \equiv t - z$; the wave amplitude is taken in all solutions to be a smooth and compact scalar function $\mathcal{A} = \mathcal{A}(\tau)$. The Brinkmann gauge is complementary to the TT gauge in the sense that it confines waves to the time and longitudinal components of the metric perturbation. The wave will therefore alter the definition of the unit timelike vector from Chapter 4, which defines the foliation[3], and unit spacelike vector l_i which defines the direction of travel, but not the polarisation vector $\varepsilon_{\bar{i}}$. Restricting to the case of weak waves, we will then have $n_i \equiv h_i{}^t/\sqrt{|g^{tt}|} = (\boldsymbol{e}^t)_i + \mathcal{O}(\mathcal{A})$, $l_i \equiv h_i{}^z/\sqrt{|g^{zz}|} = (\boldsymbol{e}^z)_i + \mathcal{O}(\mathcal{A})$ and $\varepsilon_{\bar{i}} \equiv \cos(\theta)(\boldsymbol{e}^x)_i + \sin(\theta)(\boldsymbol{e}^y)_i$. While the Brinkmann gauge is opposed to the TT gauge at the level of the metric, this turns out to be somewhat reversed at the level of the field strengths. Accordingly, it is useful to define the 'TT symmetric-traceless' operation on the indices of a general TT tensor $X_{\overline{ij}}$ as $X_{\langle\!\langle\overline{ij}\rangle\!\rangle} \equiv X_{(\overline{ij})} - \frac{1}{2}X^{\bar{k}}{}_{\bar{k}}(\eta_{\overline{ij}} - l_{\bar{i}}l_{\bar{j}})$, where we recall the original 'symmetric-traceless' operator $X_{\langle\overline{ij}\rangle} \equiv X_{(\overline{ij})} - \frac{1}{3}X^{\bar{k}}{}_{\bar{k}}\eta_{\overline{ij}}$ from Chapter 4.

The first new exact solution describes a wave in the Riemann–Cartan curvature, with vanishing torsion and two D.o.F quantified by a polarisation vector. It has the components

$$\begin{aligned} \underline{\mathcal{R}}_{\langle\overline{ij}\rangle} &= \mathcal{A}\varepsilon_{\langle\!\langle\bar{i}}\varepsilon_{\bar{j}\rangle\!\rangle} + \mathcal{O}(\mathcal{A}^2), & \mathcal{R}_{\perp\langle\overline{ij}\rangle\perp} &= \mathcal{A}\varepsilon_{\langle\!\langle\bar{i}}\varepsilon_{\bar{j}\rangle\!\rangle} + \mathcal{O}(\mathcal{A}^2), \\ {}^{\mathrm{T}}\mathcal{R}_{\perp\overline{ijk}} &= \mathcal{A}\varepsilon_{\langle\!\langle\bar{k}}\varepsilon_{[\bar{i}\rangle\!\rangle}l_{\bar{j}]} + \mathcal{O}(\mathcal{A}^2), & {}^{\mathrm{T}}\mathcal{R}_{\overline{ijk}\perp} &= -\mathcal{A}\varepsilon_{\langle\!\langle\bar{k}}\varepsilon_{[\bar{i}\rangle\!\rangle}l_{\bar{j}]} + \mathcal{O}(\mathcal{A}^2). \end{aligned} \tag{5.24}$$

These components turn out to be identical to those of the Riemann tensor in the presence of the vacuum pp-waves known from GR, and accordingly the solution (5.24) will be of special interest. By choosing to retain this solution, we forgo the right to include multipliers which deactivate these Riemann–Cartan irreps, or violate any linear dependencies among them. By reference to (B.29a) the retention of this solution is thus seen to knock out a single coupling

$$\bar{\alpha}_1 = 0. \tag{5.25}$$

[3]For this reason the Brinkmann gauge will not be a natural choice for the canonical analysis.

We note however, that the nonlinear canonical analysis of Section 5.1 suggests rather more than the two massless D.o.F of the linear theory, and so we will not be surprised to encounter further exact solutions. Accordingly, two waves of pure torsion can be found

$$ {}^{\mathrm{T}}\mathcal{T}_{\overline{ijk}} = \mathcal{A}\varepsilon_{\langle\!\langle\bar{k}}\varepsilon_{[\bar{i}\rangle\!\rangle}l_{\bar{j}]}, \quad \mathcal{T}_{\langle\overline{ij}\rangle\perp} = 4\mathcal{A}\varepsilon_{\langle\!\langle\bar{i}}\varepsilon_{\bar{j}\rangle\!\rangle}, \tag{5.26} $$

$$ {}^{\mathrm{P}}\mathcal{T} = 6\mathcal{A}, \quad \mathcal{T}_{\perp\overline{ij}} = \mathcal{A}\epsilon_{\overline{ijk}\perp}l^{\bar{k}}, \quad \mathcal{T}_{[\overline{ij}]\perp} = \mathcal{A}\epsilon_{\overline{ijk}\perp}l^{\bar{k}}, \tag{5.27} $$

where separate solutions (5.26) and (5.27) encode two and one massless D.o.F., respectively. The torsion waves themselves are accompanied by further excitations of the Riemann–Cartan tensor. In the case of (5.26), this excitation is precisely of the same form as in (5.24), while (5.27) invokes a curvature wave of a kind not found in GR. The solutions Eqs. (5.24), (5.26) and (5.27) are essentially independent, though further cross-terms are activated in the Riemann–Cartan tensor when torsion waves of type (5.26) and (5.27) interfere. We are less interested in preserving these solutions, since they would appear to contribute unobserved radiative D.o.F to the theory, and so we derive from Eqs. (5.26) and (5.27) no new limits on the multipliers. It is worth mentioning that since the Riemann–Cartan and torsion tensors are generally covariant quantities: the range of irreps activated by any solution is independent of our gauge choice, Brinkmann or otherwise.

Having analysed the massless solutions, we turn to our previous treatment of the cosmology in which the background was described by three variables: the Hubble number H, and scalars ψ and ϕ. In the general PGT$^{\mathrm{q},+}$ the SO(3) irreps involved with the background are

$$ \begin{gathered} \underline{\mathcal{R}} = \frac{3}{2}(\psi^2-\phi^2), \quad \mathcal{R}_{\perp\perp} = \frac{3}{2}(\dot{\phi}-H\phi), \quad {}^{\mathrm{P}}\mathcal{R}_{\perp\circ} = -3\phi\psi, \quad {}^{\mathrm{P}}\mathcal{R}_{\circ\perp} = -3(\dot{\psi}+H\psi), \\ {}^{\mathrm{P}}\mathcal{T} = 6\psi, \quad \mathcal{T}^{\bar{k}}{}_{\bar{k}\perp} = -\frac{3}{2}(2H+\phi). \end{gathered} \tag{5.28} $$

We have of course provided in Chapter 3 a complete statement of the dynamics which govern these background fields, and obtained in the context of Case 2 and Case 16 the CS, which we believe to describe the background of the late Universe. Since we recall from Chapter 2 the importance of complicated deviations from the CS in the early Universe, we will not confine ourselves to any given solution. No extra relations are therefore assumed among the irreps of (5.28), and a scan of Eqs. (B.29b) to (B.29i) reveals that this has a devistating impact on the remaining multipliers. Building on our condition $\bar{\alpha}_1 = 0$ from (5.24), all other irreducible components of the $\lambda_{ij}{}^{kl}$ multiplier under $\mathrm{SO}^+(1,3)$ are eliminated, other than the Faraday part ${}^5\lambda_{ij}$ as controlled by $\bar{\alpha}_5$. In turn, this multiplier disables part of the Riemann–Cartan tensor $\mathcal{R}_{[ij]}$, which (incidentally) vanishes in any Riemannian theory, $R_{[\mu\nu]} \equiv 0$. A similar effect occurs within the torsion sector, whose multiplier $\lambda_i{}^{jk}$ is reduced to its 16 tensor parts ${}^1\lambda^i{}_{jk}$, as controlled by $\bar{\beta}_1$. The resulting coupling constraints are thus added to (5.25)

$$ \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = \bar{\alpha}_6 = \bar{\beta}_2 = \bar{\beta}_3 = 0. \tag{5.29} $$

5.2.4 The torsionless vacuum

In Section 5.1 it was argued that we should be willing to cede the unitarity and renormalisability of our linearised theory. We will now show that such a sacrifice is actually demanded by the phenomenologically acceptable multipliers identified in Section 5.2.3. We examine the field equations of the theory in the regime of weak gravity, which was previously considered in Sections 1.2.1 and 2.3. It is natural to transition to the second-order formalism, in which the rotational gauge field is decomposed into the Ricci rotation coefficients $c^i{}_{jk}$ and the contortion $\mathcal{K}_{ij}{}^k$

$$A^{ij}{}_\mu \equiv \frac{1}{2}\big(2c^{[ij]}{}_k - c_k{}^{ij}\big)b^k{}_\mu + \mathcal{K}^{ij}{}_k b^k{}_\mu, \quad c^i{}_{jk} \equiv 2h_{[j|}{}^\mu h_{|k]}{}^\nu \partial_\mu b^i{}_\nu, \quad \mathcal{K}_{ijk} \equiv -\frac{1}{2}\big(2\mathcal{T}_{[ij]k} - \mathcal{T}_{kij}\big). \tag{5.30}$$

Since the last equality in (5.30) is invertible, this decomposition separates out the torsion algebraically without any loss of generality.

We will carry over our choice of Cartesian coordinates and Lorentz vectors from (5.23). The rotational gauge field is assumed to be perturbative. This is made consistent with (5.30) via the perturbation scheme (2.32) for the translational gauge field, which was used also in [152, 153]. We will not work with the full irreducible decomposition of the translational gauge field perturbation, preferring the variables presented already in (2.33) by analogy to linearised metrical gravity. Indeed, the remaining Poincaré gauge freedoms may then be fixed with the familiar harmonic gauge choice $\mathfrak{a}_{ij} = \partial_i \bar{\mathfrak{s}}_j{}^i = 0$. We take the contortion and the geometric multipliers all to be $\mathcal{O}(f)$. The full $\mathrm{SO}^+(1,3)$ decomposition becomes a worthwhile investment when manipulating these fields, and we will label the constituent parts according to Eqs. (C.3a) and (C.3b). A natural consequence of the second-order formalism in combination with quadratic gravity is that we will be forced to deal with *fourth-order* field equations. In order to minimise the explicit use of derivatives, we define the Faraday tensors associated with the vector and pseudovector parts of the torsion ${}^2\mathcal{F}_{ij} \equiv \partial_i {}^2\mathcal{T}_j - \partial_j {}^2\mathcal{T}_i$ and ${}^3\mathcal{F}_{ij} \equiv \partial_i {}^3\mathcal{T}_j - \partial_j {}^3\mathcal{T}_i$. When examining the field equations, we need only admit the two multipliers remaining from Eqs. (5.25) and (5.29). The linearised rotational gauge field equation (5.15b) then decomposes into its irreducible parts

$$\begin{aligned} {}^1\sigma_{i[jk]} = &-\frac{1}{3}\eta_{i[j}{}^2\sigma_{k]} + \frac{4(\hat{\alpha}_5-\hat{\alpha}_6)}{3}\partial_i {}^2\mathcal{F}_{jk} - \frac{2(\hat{\alpha}_5+2\hat{\alpha}_6)}{3}\big(\epsilon_{i[j|lm}\partial_{|k]} - \epsilon_{jklm}\partial_i\big){}^3\mathcal{F}^{lm} \\ &- \frac{8(\hat{\alpha}_5+2\hat{\alpha}_6)}{9}\partial_l\big(2\partial_i {}^1\mathcal{T}^l{}_{[jk]} + \partial_j {}^1\mathcal{T}^l{}_{[ik]} - \partial_k {}^1\mathcal{T}^l{}_{[ij]}\big) + 24\hat{\alpha}_6\partial_l\partial_{[j}{}^1\mathcal{T}_{k]i}{}^l \\ &- 8\hat{\alpha}_6\Box\big(\partial_{[j|}\bar{\mathfrak{s}}_{i|k]} + 3\eta_{i[j}\partial_{k]}\bar{\mathfrak{s}}\big) + \frac{8\bar{\beta}_1}{3}{}^1\lambda_{i[jk]} + \frac{4\bar{\alpha}_5}{3}\big(\partial_{[j|}{}^5\lambda_{i|k]} + \partial_i {}^5\lambda_{jk}\big), \end{aligned} \tag{5.31a}$$

$${}^2\sigma_i = \frac{8(\hat{\alpha}_5-\hat{\alpha}_6)}{3}\big(\partial_l {}^2\mathcal{F}^l{}_i - 2\partial^j\partial^k {}^1\mathcal{T}_{j[ki]}\big) - 4m_{\mathrm{p}}{}^2\hat{\beta}_2 {}^2\mathcal{T}_i + 4\bar{\alpha}_5\partial_l {}^5\lambda^l{}_i, \tag{5.31b}$$

$$\begin{aligned} {}^3\sigma_i = &\frac{8(\hat{\alpha}_5+2\hat{\alpha}_6)}{3}\big(3\partial_l {}^3\mathcal{F}^l{}_i + 2\epsilon_{ijkl}\partial_m\partial^j {}^1\mathcal{T}^{mkl}\big) + 12(\hat{\alpha}_3+2\hat{\alpha}_6)\partial_i\partial_l {}^3\mathcal{T}^l \\ &- 48\hat{\beta}_3 m_{\mathrm{p}}{}^2 {}^3\mathcal{T}_i - 4\bar{\alpha}_5\epsilon_{ijkl}\partial^j {}^5\lambda^{kl}. \end{aligned} \tag{5.31c}$$

The linearised translational gauge field equation (5.15a), without being decomposed, is

$$\tau_{ij} = -\frac{4\hat{\beta}_2}{3}\big(\partial_j {}^2\mathcal{T}_i - \eta_{ij}\partial_l {}^2\mathcal{T}^l\big) - 2\hat{\beta}_3\epsilon_{ijkl}{}^3\mathcal{F}^{kl} - \frac{8\bar{\beta}_1}{3}\partial^l {}^1\lambda_{j[li]}, \tag{5.32}$$

and by comparing (5.32) with the divergence of (5.31b) we finally connect with the property first mentioned in Section 2.5.4: both the matter and gravitational SET must be strictly *traceless* $\tau \equiv \tau^i{}_i = 0$. This is not too surprising given that the linearisations of Case 2 and Case 16 are required to be renormalisable, and the naturally suggested geometric multipliers from Eqs. (5.25) and (5.29) do not change this feature.

In order to connect with any Newtonian limit, we would prefer τ_{ij} to act as a dynamical source for $\bar{\mathfrak{s}}_{ij}$, which will require the elimination of both torsion and multipliers from (5.32). We can perform this elimination by combining (5.32) with the divergences of Eqs. (5.31a) to (5.31c), in order to obtain the linearised Belinfante tensor $\tau_{\mathrm{B}\ ij} \equiv \tau_{ij} - \frac{1}{2}\partial_k\left(2\sigma_{(ij)}{}^k - \sigma^k{}_{ji}\right)$, which is symmetric by construction. Implementing this, we obtain

$$\tau_{\mathrm{B}\ ij} = 16\hat{\alpha}_6\partial^k\partial^l\partial_{(i|}{}^1\mathcal{T}_{k[|j)l]} - 8\hat{\alpha}_6\Box\partial_l\left({}^1\mathcal{T}_{ij}{}^l - {}^1\mathcal{T}^l_{(ij)}\right) - 4\hat{\alpha}_6\Box\left(3\Box\bar{\mathfrak{s}}_{ij} - (\eta_{ij}\Box - \partial_i\partial_j)\bar{\mathfrak{s}}\right). \tag{5.33}$$

The final term in (5.33) looks far more promising, but the Newtonian limit is still not clear because in Case 2 and Case 16 we are not be able to eliminate the remaining tensor part of the torsion ${}^1\mathcal{T}^i{}_{jk}$, whose third derivatives pollute the expression. The geometric multipliers however allow us to do precisely this. The multiplier equations (5.19) are

$$\bar{\alpha}_5\left(2\partial_l{}^1\mathcal{T}^l_{[ij]} - 4^2\mathcal{F}_{ij} + 3\epsilon_{ijkl}{}^3\mathcal{F}^{kl}\right) = \bar{\beta}_1{}^1\mathcal{T}_{i[jk]} = 0, \tag{5.34}$$

so it is clear that we should concretely introduce our first geometric multiplier ${}^1\lambda^i{}_{jk}$ by setting $\bar{\beta}_1 \neq 0$. As always, we will confine ourselves to matter in which the macroscopic spin is vanishing (upon averaging and in detail), so that the linearised source currents will obey

$$\tau_{ij} = \tau_{(ij)} = \tau_{\mathrm{B}\ ij}, \quad \sigma^k{}_{ij} = 0. \tag{5.35}$$

The torsion now entirely detaches from the stress-energy equation, which reads simply

$$\tau_{ij} = -4\hat{\alpha}_6\Box\left(3\Box\bar{\mathfrak{s}}_{ij} - (\eta_{ij}\Box - \partial_i\partial_j)\bar{\mathfrak{s}}\right), \tag{5.36}$$

and the traceless fourth-order equation in (5.36) is the weak limit of conformal gravity (CG) [57].

The CG theory follows from the following metrical action on $\mathcal{M}$

$$S_{\mathrm{T}} = \int \mathrm{d}^4x\sqrt{-g}\left(\alpha_{\mathrm{CG}}W_{\rho\sigma\mu\nu}W^{\rho\sigma\mu\nu} + L_{\mathrm{M}}\right), \tag{5.37}$$

where the Weyl tensor in the Riemannian spacetime V_4 follows an analogous definition to that in (A.26)

$$W_{\rho\sigma\mu\nu} \equiv R_{\rho\sigma\mu\nu} - \frac{1}{2}\left(g_{\rho\mu}R_{\sigma\nu} - g_{\rho\nu}R_{\sigma\mu} - g_{\sigma\mu}R_{\rho\nu} + g_{\sigma\nu}R_{\rho\mu}\right) + \frac{1}{6}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right)R, \tag{5.38}$$

and α_{CG} is a dimensionless coupling. Our conventions for metrical quantities were given previously in (1.7). The CG field equations are obtained with a metrical variation

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}}, \quad T_{\mu\nu} = -4\alpha_{\mathrm{CG}}\left(2\nabla^\rho\nabla^\sigma + R^{\rho\sigma}\right)W_{\mu\rho\nu\sigma}, \tag{5.39}$$

and under the linearisation scheme set out in Eqs. (1.9) and (1.11) we find that (5.39) becomes

$$\check{T}_{\mu\nu} = \frac{2}{3}\alpha_{\mathrm{CG}}\check{\Box}\left(3\check{\Box}\bar{h}_{\mu\nu} - (\check{g}_{\mu\nu}\check{\Box} - \check{\nabla}_\mu\check{\nabla}_\nu)\bar{h}\right). \tag{5.40}$$

Comparing (5.40) to (5.36) and recalling that $h_{\mu\nu}$ is the analogue to $2\mathfrak{s}_{ij}$, we find

$$\hat{\alpha}_6 \overset{\text{an}}{=} -\frac{1}{3}\alpha_{\text{CG}}. \tag{5.41}$$

The linearised matter sector is required to be conformally invariant, and so the isolated field equation (5.36) can in principle be derived from a *renormalisable* action. The particle spectrum of (5.36) comprises six D.o.F: the graviton and a massless vector, along with a massless spin-two ghost [58]. The presence of this ghost – a null wave on the Minkowski vacuum whose amplitude increases linearly in time, and a classical instability – is expected in linearised, fourth-order theories such as (5.37). It appears now as a direct consequence of the ${}^1\lambda^i{}_{jk}$ multiplier.

The geometric multiplier ${}^1\lambda^i{}_{jk}$ suppresses the tensor part ${}^1\mathcal{T}^i_{jk}$ of the torsion, but what about the vector ${}^2\mathcal{T}^i$ and axial vector ${}^3\mathcal{T}^i$ parts whose dynamics are encoded in Eqs. (5.31a) to (5.31c)? It is clear from the outset that (5.31a) serves only to determine ${}^1\lambda^i{}_{jk}$ algebraically; it has no contractions and *no further physical content*. Turning then to Eqs. (5.31b) and (5.31c), we start with the conservative assumption that $\bar{\alpha}_5 = 0$, in which case these equations are independent. Recalling the vanishing spin condition in (5.35), we see that the vector theory (5.31b) can be derived separately from a unitarity Proca action so long as

$$(\hat{\alpha}_5 - \hat{\alpha}_6)\hat{\beta}_2 < 0, \tag{5.42}$$

which happens to be compatible with the original unitarity condition (5.6). The axial vector described by (5.31c) poses more of a problem: since $\hat{\alpha}_6 < 0$ and $\hat{\alpha}_3 = 0$ are required by (5.3), we see that its field equation will never derive from an isolated action with a purely Maxwellian kinetic term: it must always contain an Ostrogradsky ghost in its helicity-zero part [188].

It is the presence of this ghost which suggests us to consider $\bar{\alpha}_5 \neq 0$, and so introduce our second geometric multiplier ${}^5\lambda_{ij}$. This is the Faraday multiplier of the Riemann–Cartan tensor; it has no effect on the Weyl stress-energy relation in (5.36), nor will it change our interpretation of (5.31a) as an algebraic formula for the torsion multiplier ${}^1\lambda^i{}_{jk}$. However, its appearance in Eqs. (5.31b) and (5.31c) prevents these equations from dynamically determining the vector and axial vector torsion. In order to discover the dynamics of the torsion sector, we refer instead to the first equality in (5.34), which tells us that these remaining vector potentials are no longer independent: their Faraday tensors are dual to each other. Thus the magnetic field derived from the vector torsion will be the electrical field derived from the axial vector, and vice versa. By taking the divergences of the first expression in (5.34), and of its dual, we see that both remaining parts of the torsion describe a single massless vector $\partial_l {}^2\mathcal{F}^l{}_i = \partial_l {}^3\mathcal{F}^l{}_i = 0$, where $4{}^2\mathcal{F}_{ij} = 3\epsilon_{ijkl}{}^3\mathcal{F}^{kl}$. Superficially, the massless vector seems a more appealing unified descriptor of the torsion sector, since it can be viewed as following from a renormalisable and unitary theory.

The validity of applying the PCR criterion piecewise to the field equations is not, however, clear. As with determining the unitarity of the S-matrix, such analyses should be performed very carefully, beginning at the level of the action. Indeed it is easy to see that the picture is not so simple as 'adding' the individual theories: the CG field equations in (5.36) do not contain the Maxwell SET of the independent torsion vector. The key observation in this secion is therefore that the torsion multiplier ${}^1\lambda^i{}_{jk}$ produces a classical ghost which renders the Minkowski vacuum *unstable*.

5.2.5 The CS vacuum

A further objection which now arises out of (5.36) is the difficulty in reconciling Bach's CG with a sensible Newtonian limit [59]. However, our discussion so far in Section 5.2.4 was based around the same flat, torsion-free background as assumed in [152, 153], and we found in Chapters 2 and 3 that the relative vacuum of the background cosmology does not adopt this shape: the torsion psudoscalar Q (or ψ) and possibly also the scalar U (or ϕ) typically adopt the *finite* values of the CS. Part of the Riemannian curvature will also be active through the Hubble number in any realistic cosmology. The hierarchical model of structure formation suggests however that H should not be of primary importance; we prefer any Newtonian limit not to rely on an expanding Universe, but to remain valid even deep within collapsing overdensities.

Let us now consider a linearisation around the CS, on an energy scale far exceeding the Hubble number and corresponding to some *astrophysical* length scale. Such a limit could also be found at any scale in the sufficiently late Universe of Class ^{3}C* with $\Lambda_{\mathrm{b}} = 0$, or Class ^{2}A* with a finely tuned *negative* bare cosmological constant $\Lambda_{\mathrm{b}} = 2\hat{\beta}_3 {m_{\mathrm{p}}}^2/\hat{\alpha}_6$; in these cases the matter will dilute away and a static universe is approached very slowly in cosmic time. We will work with the former case, setting $\hat{\beta}_3 = 0$ in (5.3) – owing to its interpretation as the cosmological constant, this coupling could equally be neglected on perturbative grounds. The CS background will be a flat spacetime containing a spin and matter vacuum, with the constant torsion $Q_\infty = \pm m_{\mathrm{p}}/\sqrt{-3\hat{\alpha}_6}$ and $U_\infty = 0$ (geometrically interpreted as a Weitzenböck T_4 space). Any lingering Hubble flow will define a convenient foliation n_i for the ADM formalism, though we must set $H_\infty = 0$; we shall also assume $k = 0$, though we know from Section 2.5.3 that the background dynamics are indifferent to spatial curvature. Perturbation theory is not straightforward even around this simplest of CS vacua, and so we shall confine ourselves to a static Newtonian scalar potential $\partial_\perp \varphi = 0$, with the translational gauge field perturbation[4]

$$\mathfrak{s}_{ij} = (2n_i n_j - \eta_{ij})\varphi, \quad \bar{\mathfrak{s}}_{ij} = 2n_i n_j \varphi, \quad \mathfrak{s} = -\bar{\mathfrak{s}} = -2\varphi. \tag{5.43}$$

Note that φ in (5.43) is precisely the same quantity we encountered in Section 1.2.4, and extended nonlinearly for GR and ECT throughout Chapter 1. We also employ our potential in constructing the ansätze for the torsion perturbations, in which case the vector and axial vector can only be

$${}^2\mathcal{T}_i = \zeta_0 m_{\mathrm{p}} n_i \varphi + \zeta_1 \partial_i \varphi, \quad {}^3\mathcal{T}_i = \pm \frac{m_{\mathrm{p}}}{\sqrt{-3\hat{\alpha}_6}} n_i + \xi_0 m_{\mathrm{p}} n_i \varphi + \xi_1 \partial_i \varphi, \tag{5.44}$$

for dimensionless constants ζ_0, ζ_1, ξ_0 and ξ_1. We need not build the (purely perturbative) tensor ansatz for the torsion ${}^1\mathcal{T}^i{}_{jk}$ since it will be switched off by its own multiplier ${}^1\lambda^i{}_{jk}$; the selfsame multiplier will then be determined by (5.31a), however it appears in the new vacuum. The Faraday Riemann–Cartan multiplier requires closer attention, and its most general form will be

$${}^5\lambda_{ij} = \chi_0 n_{[i}\partial_{j]}\varphi - \chi_1 \epsilon_{ijk\perp}\partial^k \varphi, \tag{5.45}$$

for further constants χ_0 and χ_1. The remaining rotational gauge field and Faraday multiplier Eqs. (5.31b), (5.31c) and (5.34) now become

$${}^2\sigma_i = -4\big(1 \mp \sqrt{-3\hat{\alpha}_6}\xi_0\big) m_{\mathrm{p}} \partial_i \varphi - \frac{2}{3}\big(3\bar{\alpha}_5 \chi_0 - 4(\hat{\alpha}_5 - \hat{\alpha}_6)\zeta_0\big) n_i \Box\varphi, \tag{5.46}$$

[4] Note that φ is assumed to vanish suitably at spatial infinity.

$$
\begin{aligned}
{}^3\sigma_i = & \frac{8}{\sqrt{-3\hat{\alpha}_6}}\Big[\pm\big(\hat{\alpha}_5 - 2\hat{\alpha}_6(2+\zeta_1)\big) + \sqrt{-3\hat{\alpha}_6}(\bar{\alpha}_5\chi_1 + (\hat{\alpha}_5 + 2\hat{\alpha}_6)\xi_0)\Big] m_\mathrm{p} n_i \Box\varphi \\
& + 24\hat{\alpha}_6\xi_1\partial_i\Box\varphi,
\end{aligned} \tag{5.47}
$$

$$
0 = \bar{\alpha}_5\left[\frac{2}{3}\zeta_0 n_{[i}\partial_{j]}\varphi \pm \frac{1}{2\sqrt{-3\hat{\alpha}_6}}\big(1 \pm \sqrt{-3\hat{\alpha}_6}\xi_0\big)\epsilon_{ijk\perp}\partial^k\varphi\right]. \tag{5.48}
$$

In the spin-free case (5.35), the system Eqs. (5.46) to (5.48) is only soluble if we set

$$
\bar{\alpha}_5 = 0, \tag{5.49}
$$

and so we conclude that in the presence of the tensor torsion multiplier, the Faraday Riemann–Cartan multiplier is ultimately *inconsistent* with the static Newtonian potential. Once (5.49) is imposed, the system has a *unique* solution

$$
\zeta_0 = \xi_1 = 0, \quad \zeta_1 = \frac{\hat{\alpha}_5}{\hat{\alpha}_6} - 1, \quad \xi_0 = \pm\frac{1}{\sqrt{-3\hat{\alpha}_6}}. \tag{5.50}
$$

We are now in a position to consider the translational gauge field or stress-energy equation, where any sensible Newtonian limit should become manifest. We will not follow the formal Rosenfeld procedure for generalising the Belinfante SET to the CS vacuum, but directly consider the linearisation of (5.15a). After substituting for all parts of the torsion, the multipliers and the solution (5.50), the spin-free stress-energy equation finally becomes

$$
\tau_{ij} = -8\hat{\alpha}_6\Box\big((3n_i n_j - \eta_{ij})\Box + \partial_i\partial_j\big)\varphi - \frac{8}{3}\Big(\frac{\hat{\alpha}_5}{\hat{\alpha}_6} - 1\Big){m_\mathrm{p}}^2 n_i n_j \Box\varphi. \tag{5.51}
$$

The first three fourth-order terms in (5.51) are readily identifiable as the conformal Weyl operator from (5.36), as it acts on the Newtonian potential. The final second-order term breaks the conformal symmetry, and emerges naturally as a consequence of perturbative mixing with the CS vacuum. This term may be compared with (1.20) in Chapter 1: among the various possible forms, its structure is precisely that of Newtonian gravity. By thus identifying one of our two remaining couplings with a numerically natural value

$$
\hat{\alpha}_5 = \frac{7}{4}\hat{\alpha}_6, \tag{5.52}
$$

our theory anticipates Newton's law of universal gravitation.

By setting $\bar{\alpha}_5 = 0$ in (5.49), we will lose any quantum mechanical benefits imparted by the multiplier on the Minkowski vacuum in Section 5.2.4. Indeed both ${}^2\mathcal{T}^i$ and ${}^3\mathcal{T}^i$ will contain even classical ghosts, and the torsion will feel a tree-level instability on the Minkowski vacuum. We recall however that the addition of cosmological matter was shown in Chapters 2 and 3 to have much the same result, with that instability leading instead back to the CS vacuum. Notwithstanding the multiplier-induced torsion instability of the Minkowski vacuum, we note that (5.52) in combination with (5.3) is at least consistent with the original Minkowskian unitarity condition (5.6), as it holds in the total absence of multipliers $\bar{\beta}_1 = \bar{\alpha}_5 = 0$. On these grounds, we will retain our new condition in the final formulation of the theory.

The unresolved feature of (5.51) is now the survival of the fourth-order Weyl operator. Assuming however as in Section 2.6 that $\hat{\alpha}_6$ is $\mathcal{O}(1)$, the remaining fourth-order terms need not endanger the Newtonian limit until the Riemannian curvature approaches the inverse square Planck length[5].

5.3 The Hamiltonian picture

5.3.1 The new super-Hamiltonian

Having briefly toured the Lagrangian formulation of geometric multipliers in Section 5.2, we now turn to the Hamiltonian formulation. Our total Hamiltonian from (4.13) must now be extended to

$$\mathcal{H}_{\mathrm{T}} \equiv \mathcal{H}_{\mathrm{C}} + u^k{}_0\varphi_k{}^0 + \frac{1}{2}u^k{}_0\varphi_k{}^0 + (u\cdot\varphi) + u_i{}^{kl}\varphi^i{}_{kl} + u_{ij}{}^{kl}\varphi^{ij}{}_{kl}, \tag{5.53}$$

where the primary constraints

$$\varphi^{ij}{}_{kl} \equiv \pi^{ij}{}_{kl} \approx 0, \quad \varphi^{i}{}_{kl} \equiv \pi^{i}{}_{kl} \approx 0, \tag{5.54}$$

must be introduced because the Lagrangian (5.8) is independent of the multiplier velocities $\dot{\lambda}^{ij}{}_{kl}$ and $\dot{\lambda}^{i}{}_{kl}$. The PiCs take the form

$$(u\cdot\varphi) \equiv \frac{1}{32}\sum_A c^{\perp}_A \nu(\hat{\alpha}^{\perp\perp}_A)^A u_{\acute{v}}{}^A\varphi^{\acute{v}} + \frac{1}{8m_{\mathrm{p}}}\sum_E c^{\perp}_E \nu(\hat{\beta}^{\perp\perp}_E)^E u_{\acute{v}}{}^E\varphi^{\acute{v}}. \tag{5.55}$$

We note a change in (5.55) from the previous formalism in Chapter 4, in that we introduce factors of $c^{\perp}_A/32$ and $c^{\perp}_E/8$ – this just amounts to a rescaling of the ${}^A u^{\acute{v}}$ by some constants at the point of definition, and will make things more convenient in Section 5.3.2. The PiC functions are now

$${}^A\varphi_{\acute{v}} \equiv \frac{1}{J}{}^A\hat{\pi}_{\acute{v}} - 8\bar{\alpha}^{\perp\perp}_A{}^A\check{\mathcal{P}}_{\acute{v}jk}{}^{\overline{m}}\lambda^{jk}{}_{\perp\overline{m}} - 4{}^A\check{\mathcal{P}}_{\acute{v}jk}{}^{\overline{lm}}\Big(\bar{\alpha}^{\perp\|}_A\lambda^{jk}{}_{\overline{lm}} + 2\hat{\alpha}^{\perp\|}_A\mathcal{R}^{jk}{}_{\overline{lm}}\Big), \tag{5.56a}$$

$${}^E\varphi_{\acute{v}} \equiv \frac{1}{J}{}^E\hat{\pi}_{\acute{v}} - 4{m_{\mathrm{p}}}^2\bar{\beta}^{\perp\perp}_E{}^E\check{\mathcal{P}}_{\acute{v}j}{}^{\overline{m}}\lambda^{j}{}_{\perp\overline{m}} - 2{m_{\mathrm{p}}}^2{}^E\check{\mathcal{P}}_{\acute{v}j}{}^{\overline{lm}}\Big(\bar{\beta}^{\perp\|}_E\lambda^{j}{}_{\overline{lm}} + 2\hat{\beta}^{\perp\|}_E\mathcal{T}^{j}{}_{\overline{lm}}\Big), \tag{5.56b}$$

so they generally acquire a dependency on the multiplier fields. Within the canonical Hamiltonian defined in (4.14), the super-Hamiltonian in (4.15a) is modified beyond the formula in (4.17) to

$$\begin{aligned}\mathcal{H}_{\perp} \equiv\ & \frac{J}{64}\sum_A c^{\perp}_A \mu(\hat{\alpha}^{\perp\perp}_A)^A\varphi_{\acute{v}}{}^A\varphi^{\acute{v}} + \frac{J}{16m_{\mathrm{p}}}\sum_E c^{\perp}_E \mu(\hat{\beta}^{\perp\perp}_E)^E\varphi_{\acute{v}}{}^E\varphi^{\acute{v}} \\ & - J\sum_I\Big(\hat{\alpha}_I\mathcal{R}^{ij}{}_{\overline{kl}} + \bar{\alpha}_I\lambda^{ij}{}_{kl}\Big){}^I\hat{\mathcal{P}}_{ij}{}^{kl}{}_{nm}{}^{\overline{pq}}\mathcal{R}^{nm}{}_{\overline{pq}} \\ & - J{m_{\mathrm{p}}}^2\sum_M\Big(\hat{\beta}_M\mathcal{T}^i{}_{\overline{kl}} + \bar{\beta}_M\lambda^i{}_{kl}\Big){}^M\hat{\mathcal{P}}_i{}^{kl}{}_n{}^{\overline{pq}}\mathcal{T}^n{}_{\overline{pq}} - n^k D_\alpha\pi_k{}^\alpha.\end{aligned} \tag{5.57}$$

The remaining parts, the linear and rotational supermomenta and the surface term, are as defined in Eqs. (4.15b) to (4.15d).

[5]Beyond the current context of asymptotically flat solutions, we speculate that the Weyl term may also be of interest in modifying the Newtonian dynamics even at astrophysical scales, fourth-order Greens functions being associated with linear potentials.

5.3.2 Consistency of geometric primaries

We will now consider the application of the Dirac–Bergmann algorithm on the general theory, and discover a substantial departure from the simple teleparallel constraint structure of (5.1). In what follows we will discuss the effects of Riemann–Cartan and torsion multipliers concurrently; while these sectors differ in certain numerical factors and notation, the discussion is essentially the same up to some placeholder results in the torsion sector. We first see that there is a new pair of secondary constraints from (5.54), $\chi^{ij}{}_{kl} \equiv \dot{\pi}^{ij}{}_{kl} \approx 0$ and $\chi^{i}{}_{kl} \equiv \dot{\pi}^{i}{}_{kl} \approx 0$, which we find to be equivalent to

$$\sum_I \bar{\alpha}_I{}^I\hat{\mathcal{P}}_{ij}{}^{kl}{}_{nm}{}^{pq}\left[8b\mathcal{R}^{nm}{}_{\overline{pq}} + \sum_A c_A^{\perp}\Big(b\mu(\hat{\alpha}_A^{\perp\perp})^A\varphi^{\acute{r}} + \nu(\hat{\alpha}_A^{\perp\perp})^A u^{\acute{r}}\Big)n_p{}^A\check{\mathcal{P}}_{\acute{r}}{}^{nm}{}_{\overline{q}}\right] \approx 0, \tag{5.58a}$$

$$\sum_M \bar{\beta}_M{}^M\hat{\mathcal{P}}_i{}^{kl}{}_n{}^{pq}\left[4m_{\text{p}}{}^2 b\mathcal{T}^n{}_{\overline{pq}} + \sum_E c_E^{\perp}\Big(b\mu(\hat{\beta}_E^{\perp\perp})^E\varphi^{\acute{r}} + \nu(\hat{\beta}_E^{\perp\perp})^E u^{\acute{r}}\Big)n_p{}^E\check{\mathcal{P}}_{\acute{r}}{}^n{}_{\overline{q}}\right] \approx 0. \tag{5.58b}$$

These secondaries correspond to two statements in the teleparallel theory, eliminating the Riemann–Cartan curvature and a multiplier. In the general theory we obtain by projections of Eqs. (5.58a) and (5.58b) up to 24 possible statements which can be written more compactly as

$$\begin{pmatrix} \bar{\alpha}_A^{\|\|\|} & \bar{\alpha}_A^{\|\perp} \\ \bar{\alpha}_A^{\perp\|} & \bar{\alpha}_A^{\perp\perp} \end{pmatrix}\begin{pmatrix} 8b^A\check{\mathcal{P}}_{\acute{l}nm}{}^{\overline{pq}}\mathcal{R}^{nm}{}_{\overline{pq}} \\ b\mu(\hat{\alpha}_A^{\perp\perp})^A\varphi_{\acute{l}} + \nu(\hat{\alpha}_A^{\perp\perp})^A u_{\acute{l}} \end{pmatrix} \approx \begin{pmatrix} \bar{\beta}_E^{\|\|\|} & \bar{\beta}_E^{\|\perp} \\ \bar{\beta}_E^{\perp\|} & \bar{\beta}_E^{\perp\perp} \end{pmatrix}\begin{pmatrix} 4m_{\text{p}}{}^2 b^E\check{\mathcal{P}}_{\acute{l}n}{}^{\overline{pq}}\mathcal{T}^n{}_{\overline{pq}} \\ b\mu(\hat{\beta}_E^{\perp\perp})^E\varphi_{\acute{l}} + \nu(\hat{\beta}_E^{\perp\perp})^E u_{\acute{l}} \end{pmatrix} \approx \mathbf{0}. \tag{5.59}$$

We note from Eq. (5.59) the counterpart in the Hamiltonian picture of the linear systems first encountered in (5.19). For any of the sectors A and E, we will next discuss the implications of these systems for various $\{\bar{\alpha}_I\}$ and $\{\bar{\beta}_M\}$, and then in each sub-case for various $\{\hat{\alpha}_I\}$ and $\{\hat{\beta}_M\}$.

Sector is not multiplier-constrained Sometimes we will have results such as

$$\bar{\alpha}_A^{\|\|\|} = \bar{\alpha}_A^{\perp\perp} = \bar{\alpha}_A^{\|\perp} = \bar{\alpha}_A^{\perp\|} = 0, \quad \bar{\beta}_E^{\|\|\|} = \bar{\beta}_E^{\perp\perp} = \bar{\beta}_E^{\|\perp} = \bar{\beta}_E^{\perp\|} = 0, \tag{5.60}$$

in which case we proceed normally, as we did in Chapter 4.

Sector is multiplier-constrained and non-singular More generally we will have

$$\bar{\alpha}_A^{\|\|\|}\bar{\alpha}_A^{\perp\perp} - \bar{\alpha}_A^{\|\perp}\bar{\alpha}_A^{\perp\|} \neq 0, \quad \bar{\beta}_E^{\|\|\|}\bar{\beta}_E^{\perp\perp} - \bar{\beta}_E^{\|\perp}\bar{\beta}_E^{\perp\|} \neq 0, \tag{5.61}$$

but not (5.60), in which case one or both of the systems (5.59) is only satisfied by a vanishing vector. In this case the first vanishing component of either system *always* gives us a secondary constraint

$${}^A\chi^{\|}_{\acute{v}} \equiv {}^A\check{\mathcal{P}}_{\acute{v}nm}{}^{\overline{pq}}\mathcal{R}^{nm}{}_{\overline{pq}} \approx 0, \quad {}^E\chi^{\|}_{\acute{v}} \equiv m_{\text{p}}{}^2\,{}^E\check{\mathcal{P}}_{\acute{v}n}{}^{\overline{pq}}\mathcal{T}^n{}_{\overline{pq}} \approx 0, \tag{5.62}$$

independently of the $\{\hat{\alpha}_I\}$ or $\{\hat{\beta}_M\}$. The parallel parts of the field strengths can then be safely eliminated from the corresponding PiC functions in Eqs. (5.56a) and (5.56b). However, we recall that this PiC function is only a PiC if $\nu(\hat{\alpha}_A^{\perp\perp}) = 1$ or $\nu(\hat{\beta}_E^{\perp\perp}) = 1$. In that case, the second vanishing component does not give us a further secondary constraint, but instead determines a multiplier

$${}^A u_{\acute{v}} \approx 0, \quad {}^E u_{\acute{v}} \approx 0, \tag{5.63}$$

and so we see that the PiC associated with sector A or E spontaneously becomes SC. If on the other hand we have $\nu(\hat{\alpha}^{\perp\perp}_{A})=0$ or $\nu(\hat{\beta}^{\perp\perp}_{E})=0$, then the vanishing of the second component means that the PiC function (with its field strength terms removed) becomes a further secondary constraint,

$$ {}^{A}\chi^{\perp}_{\acute{v}} \equiv {}^{A}\varphi_{\acute{v}} \approx 0, \quad {}^{E}\chi^{\perp}_{\acute{v}} \equiv {}^{E}\varphi_{\acute{v}} \approx 0, \tag{5.64}$$

even though it was not primarily constrained by the $\{\hat{\alpha}_{I}\}$ or $\{\hat{\beta}_{M}\}$.

We are now in a position to confirm the action of the Lorentz constraint in the Lagrangian picture. We see from the Hamiltonian E.o.M that

$$ \dot{A}^{ij}{}_{\alpha} \equiv \partial_{\alpha}A^{ij}{}_{0} + 2A^{l[j}{}_{0}A_{l}{}^{i]}{}_{\alpha} + N^{\beta}R^{ij}{}_{\beta\alpha} + \frac{\partial\sum_{A} c^{\perp}_{A}\Big(b\mu(\hat{\alpha}^{\perp\perp}_{A})^{A}\varphi_{\acute{v}} + 2\nu(\hat{\alpha}^{\perp\perp}_{A})^{A}u_{\acute{v}}\Big)^{A}\varphi^{\acute{v}}}{64\partial\pi_{ij}{}^{\alpha}}, \tag{5.65a}$$

$$ \dot{b}^{i}{}_{\alpha} \equiv \partial_{\alpha}b^{i}{}_{0} + b^{l}{}_{0}A^{i}{}_{l\alpha} - A^{i}{}_{j0}b^{j}{}_{\alpha} + N^{\beta}T^{i}{}_{\beta\alpha} + \frac{\partial\sum_{E} c^{\perp}_{E}\Big(b\mu(\hat{\beta}^{\perp\perp}_{E})^{E}\varphi_{\acute{v}} + 2\nu(\hat{\beta}^{\perp\perp}_{E})^{E}u_{\acute{v}}\Big)^{E}\varphi^{\acute{v}}}{16m_{\mathrm{p}}\partial\pi_{i}{}^{\alpha}}, \tag{5.65b}$$

and by rearranging this and projecting, we find a very useful general expression for the *velocity* parts of the Riemann–Cartan and torsion tensors in terms of canonical quantities

$$ 16b^{A}\check{\mathcal{P}}_{\acute{v}ij}{}^{\overline{k}}\mathcal{R}^{ij}{}_{\perp\overline{k}} \equiv b\mu(\hat{\alpha}^{\perp\perp}_{A})^{A}\varphi_{\acute{v}} + \nu(\hat{\alpha}^{\perp\perp}_{A})^{A}u_{\acute{v}}, \quad 8b^{E}\check{\mathcal{P}}_{\acute{v}i}{}^{\overline{k}}\mathcal{T}^{i}{}_{\perp\overline{k}} \equiv b\mu(\hat{\beta}^{\perp\perp}_{E})^{E}\varphi_{\acute{v}} + \nu(\hat{\beta}^{\perp\perp}_{E})^{E}u_{\acute{v}}. \tag{5.66}$$

Recall that these velocities are not part of the constraint algebra, and are usually found to be multipliers – we could have used this expression for example in Chapter 4. However it is now clear from (5.66) and from (5.63) and (5.64) that when the sector A or E is multiplier-constrained and non-singular, the velocity parts of the Riemann–Cartan or torsion tensors in that sector will be vanishing, no matter what is $\mu(\hat{\alpha}^{\perp\perp}_{A})$ or $\mu(\hat{\beta}^{\perp\perp}_{E})$. In combination with the canonical constraint (5.62), this means that the *whole* of the field strength tensor in the A or E sector vanishes, which is precisely the effect in (5.19) of the multipliers in the Lagrangian picture.

Sector is multiplier-constrained and singular There is a special case where neither (5.60) nor (5.61) are true. When one of the matrices is singular in this way, both consistency conditions for each A or E are equivalent but nontrivial. Once again the outcome depends on the $\{\hat{\alpha}_{I}\}$ and $\{\hat{\beta}_{M}\}$. If $\nu(\hat{\alpha}^{\perp\perp}_{A})=1$ or $\nu(\hat{\beta}^{\perp\perp}_{E})=1$, the original PiC function is indeed a PiC and

$$ {}^{A}u_{\acute{v}} \approx -\frac{8\bar{\alpha}^{\perp\|}_{A}}{\bar{\alpha}^{\perp\perp}_{A}}b^{A}\check{\mathcal{P}}_{\acute{v}nm}{}^{\overline{pq}}\mathcal{R}^{nm}{}_{\overline{pq}}, \quad {}^{E}u_{\acute{v}} \approx -\frac{4\bar{\beta}^{\perp\|}_{E}}{\bar{\beta}^{\perp\perp}_{E}}m_{\mathrm{p}}{}^{2}b^{E}\check{\mathcal{P}}_{\acute{v}n}{}^{\overline{pq}}\mathcal{T}^{n}{}_{\overline{pq}}, \tag{5.67}$$

so the PiC is again SC. In this case no new secondaries are introduced. Otherwise if $\nu(\hat{\alpha}^{\perp\perp}_{A})=0$ or $\nu(\hat{\beta}^{\perp\perp}_{E})=0$, a new secondary is introduced

$$ {}^{A}\chi^{\perp\!\!\perp}_{\acute{v}} \equiv {}^{A}\varphi_{\acute{v}} + \frac{8\bar{\alpha}^{\perp\|}_{A}\hat{\alpha}^{\perp\perp}_{A}}{\bar{\alpha}^{\perp\perp}_{A}}{}^{A}\check{\mathcal{P}}_{\acute{v}jk}{}^{\overline{lm}}\mathcal{R}^{jk}{}_{\overline{lm}} \approx 0, \quad {}^{E}\chi^{\perp\!\!\perp}_{\acute{v}} \equiv {}^{E}\varphi_{\acute{v}} + \frac{4\bar{\beta}^{\perp\|}_{E}\hat{\beta}^{\perp\perp}_{E}}{\bar{\beta}^{\perp\perp}_{E}}m_{\mathrm{p}}{}^{2}{}^{E}\check{\mathcal{P}}_{\acute{v}j}{}^{\overline{lm}}\mathcal{T}^{j}{}_{\overline{lm}} \approx 0. \tag{5.68}$$

Now again it is necessary to check the constraints from the Lagrangian picture. We see immediately from Eqs. (5.66) to (5.68) that the only such relations are

$$ \bar{\alpha}^{\|\|}_{A}{}^{A}\check{\mathcal{P}}_{\acute{v}nm}{}^{\overline{pq}}\mathcal{R}^{nm}{}_{\overline{pq}} + 2\bar{\alpha}^{\|\perp}_{A}{}^{A}\check{\mathcal{P}}_{\acute{v}nm}{}^{\overline{q}}\mathcal{R}^{nm}{}_{\perp\overline{q}} \approx 0, \quad \bar{\beta}^{\|\|}_{E}{}^{E}\check{\mathcal{P}}_{\acute{v}n}{}^{\overline{pq}}\mathcal{T}^{n}{}_{\overline{pq}} + 2\bar{\beta}^{\|\perp}_{E}{}^{E}\check{\mathcal{P}}_{\acute{v}n}{}^{\overline{q}}\mathcal{T}^{n}{}_{\perp\overline{q}} \approx 0. \tag{5.69}$$

Again, this is exactly what we expected for the singular case of (5.19).

5.3.3 Consistency of geometric secondaries

In the canonical analysis of our new general theory (5.8) we observe that the gravitational gauge fields introduce $2\times(16+24)$ canonical D.o.F, and likewise $2\times(24+36)$ D.o.F are introduced by the geometric multipliers, for a total of 200 canonical D.o.F divided over 100 fields and 100 field momenta. Typically, $2\times m$ D.o.F will have been introduced formally through m field D.o.F allocated to 'unemployed' multiplier irreps (unemployed in the sense of Eq. (5.22)). Their elimination from the final counting is equally formal, since the corresponding $\mathrm{SO}^+(1,3)$ irreps of their momenta (the primaries $\varphi^{ij}{}_{kl}$ and $\varphi^{i}{}_{kl}$ in (5.54)) will be FC. The primarily constrained momenta of 'employed' irreps are not obviously FC, since they fail to commute with their own secondaries as follows

$$\left\{\varphi^{ij}{}_{kl},{}^{A}\chi^{\perp}_{\acute{v}}\right\}\approx\left\{\varphi^{ij}{}_{kl},{}^{A}\chi^{\perp\!\!\perp}_{\acute{v}}\right\}\approx 16\Big(\bar{\alpha}^{\perp\|}_{A}\,{}^{A}\check{\mathcal{P}}_{\acute{v}}{}^{ij}{}_{\overline{kl}}+2\bar{\alpha}^{\perp\perp}_{A}n_{[k|}\,{}^{A}\check{\mathcal{P}}_{\acute{v}}{}^{ij}{}_{|\bar{l}]}\Big)\delta^{3},\tag{5.70a}$$

$$\left\{\varphi^{i}{}_{kl},{}^{E}\chi^{\perp}_{\acute{v}}\right\}\approx\left\{\varphi^{i}{}_{kl},{}^{E}\chi^{\perp\!\!\perp}_{\acute{v}}\right\}\approx 4\Big(\bar{\beta}^{\perp\|}_{E}\,{}^{E}\check{\mathcal{P}}_{\acute{v}}{}^{i}{}_{\overline{kl}}+2\bar{\beta}^{\perp\perp}_{E}n_{[k|}\,{}^{E}\check{\mathcal{P}}_{\acute{v}}{}^{i}{}_{|\bar{l}]}\Big)\delta^{3}.\tag{5.70b}$$

We note however that every J^P contains two momentum irreps, up to placeholder cases in the torsion sector, and from these parts we see that the combinations

$$2c^{\perp}_{A}\bar{\alpha}^{\perp\perp}_{A}\,{}^{A}\check{\mathcal{P}}_{\acute{u}ij}{}^{\overline{kl}}\pi^{ij}{}_{\overline{kl}}-c^{\|}_{A}\bar{\alpha}^{\perp\|}_{A}\,{}^{A}\check{\mathcal{P}}_{\acute{u}ij}{}^{\bar{l}}\pi^{ij}{}_{\perp\bar{l}}\approx 0,\quad 2c^{\perp}_{E}\bar{\beta}^{\perp\perp}_{E}\,{}^{E}\check{\mathcal{P}}_{\acute{u}i}{}^{\overline{kl}}\pi^{i}{}_{\overline{kl}}-c^{\|}_{E}\bar{\beta}^{\perp\|}_{E}\,{}^{E}\check{\mathcal{P}}_{\acute{u}i}{}^{\bar{l}}\pi^{i}{}_{\perp\bar{l}}\approx 0,\tag{5.71}$$

commute with ${}^{A}\chi^{\perp}_{\acute{v}}$ and ${}^{A}\chi^{\perp\!\!\perp}_{\acute{v}}$ or ${}^{E}\chi^{\perp}_{\acute{v}}$ and ${}^{E}\chi^{\perp\!\!\perp}_{\acute{v}}$, and are in fact FC. We will not attempt here a general theory of the remaining consistency conditions. For such a theory, the effects of the $\{\bar{\alpha}_I\}$, $\{\bar{\beta}_M\}$ must in some sense be 'multiplied' by those of the $\{\hat{\alpha}_I\}$, $\{\hat{\beta}_M\}$, and the interactions are not obvious. For our purposes therefore, the consistencies of the remaining constraints must be obtained in detail.

We also recall that we must always subtract 2×10 D.o.F due to the sPFCs, and in the nonlinear theory we assume all the 2×10 sSFCs will be independent and must also be removed. We show separately in Appendices C.6 and C.8 how the sSFCs may be reduced or become degenerate in the linear theory, if the Einstein–Hilbert term is absent.

The Einstein–Hilbert term also plays a critical rôle in preserving the typical SC nature of nonlinear if-constraints down into the linear theory, since it supports many of the linearised commutators. By examining the entries of Table 4.2 we notice that, at the linear level, removal of the Einstein–Hilbert term knocks out one of each of the 0^+ and 2^+ commutators entirely, while generally thinning the ranks of the PPM for any given set of torsion couplings $\{\hat{\beta}_M\}$. A theory formulated to have suitably few D.o.F in the linear regime, should it lack an Einstein–Hilbert term, then tends to suffer very severely from field activation: such was our impression from Chapter 4 and in Section 5.1. The problem can thus be broadly understood through the vanishing of mass parameters in Table 4.2, and this is likely attributable to the PCR requirement in [152, 153].

The geometric multipliers act to extend Table 4.2 by generating a new sector of if-constraints. This in turn produces more opportunities for linear commutators to arise, with Eqs. (5.70a) and (5.70b) representing very basic examples contained entirely within the multiplier if-constraint sector. Crucially, linear commutator interference between the multiplier and original $\mathrm{PGT}^{\mathrm{q},+}$ if-constraint sectors is naturally facilitated: parallel secondaries ${}^{A}\chi^{\|}_{\acute{v}}$, ${}^{E}\chi^{\|}_{\acute{v}}$ (which express the gravitational gauge fields via the parallel field strengths) can be added *independently* of the original theory's PiCs (which express their conjugate field momenta). In the original $\mathrm{PGT}^{\mathrm{q},+}$, the only sources of field strengths in the constraint algebra were the PiCs themselves, in which they were pre-multiplied by parameter combinations which became statistically more likely to vanish as more PiC functions were constrained by choice of couplings. A particularly useful parallel secondary (as we shall see in Section 5.3.4) is ${}^{\mathrm{T}}\underline{\chi}^{\parallel}_{\overline{klm}}$. We can see from Eqs. (4.8a) to (4.8d), this SO(3) tensor part of the canonical torsion is *forbidden* from arising in the PiCs of standard $\mathrm{PGT}^{\mathrm{q},+}$, and fortuitously produces a linear commutator with many other quantities.

Despite their utility, the new commutators tend to suffer from an old challenge (noticed for example in [226]) as follows. While the parallel field strengths do express the fields conjugate to the field momenta, they also contain spatial gradients of those fields. Within the formal definition of the Poisson bracket, this can lead to gradients of the equal-time Dirac function, and an apparent loss of explicit covariance for the more complex expressions. In Appendix C.9 we clarify such situations by constructing a general and covariant expression for the Poisson bracket, which then takes the form of a differential operator.

We note finally that the mechanism outlined above *only* works because the new fields in Eq. (5.8) are multipliers: propagating D.o.F would tend to increase with any new kinetic terms, offsetting any benefits.

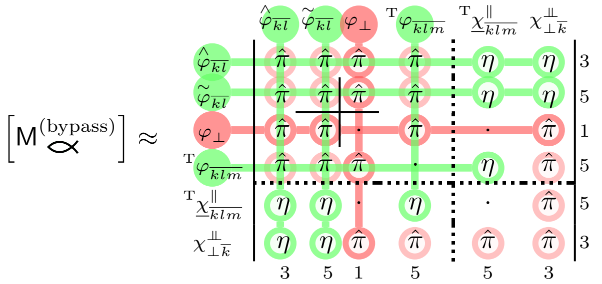
5.3.4 Case 16 with tensor bypass

In this final section, we will demonstrate the multiplier constraint structure on the unique phenomenologically viable theory arrived at in Section 5.2. The 16 'tensor part' components of the torsion are *bypassed* with multiplier fields, which can be eliminated in the final field equations. Under these multipliers the Riemann–Cartan tensor is not constrained, while the torsion must obey Eq. (B.29g), which amounts to singular constraints on the vector 1^+ and 1^- sectors, and nonsingular constraints on the tensor 2^+ and 2^- sectors. For simplicity as in Section 5.1 we again apply the tensor bypass to Case 16 rather than Case 2. In particular we note alterations to two PiCs, and new singular and parallel secondaries

$$
\begin{aligned}
&\overset{\wedge}{\varphi}_{\overline{ij}} = \frac{1}{J}\overset{\wedge}{\hat{\pi}}_{\overline{ij}} - \frac{2\bar{\beta}_1}{3}{m_{\mathrm{p}}}^2\left(\lambda_{\perp\overline{ij}} - \lambda_{[\overline{ij}]\perp}\right) \approx 0, \quad \overset{\sim}{\varphi}_{\overline{ij}} = \frac{1}{J}\overset{\sim}{\hat{\pi}}_{\overline{ij}} + 2\bar{\beta}_1{m_{\mathrm{p}}}^2\lambda_{\langle\overline{ij}\rangle\perp} \approx 0, \\
&\chi^{\perp\!\!\perp}_{\perp\bar{i}} = \frac{1}{J}\hat{\pi}_{\perp\bar{i}} + 2\hat{\beta}_2{m_{\mathrm{p}}}^2\overset{\rightharpoonup}{\mathcal{T}}_{\bar{i}} - \frac{2\bar{\beta}_1}{3}{m_{\mathrm{p}}}^2\overset{\rightharpoonup}{\lambda}_{\bar{i}} + \frac{4\bar{\beta}_1}{3}{m_{\mathrm{p}}}^2\lambda_{\perp\bar{i}\perp} \approx 0, \quad {}^{\mathrm{T}}\underline{\chi}^{\parallel}_{\overline{ijk}} = {}^{\mathrm{T}}\mathcal{T}_{\overline{ijk}} \approx 0.
\end{aligned}
\tag{5.72}
$$

We take the multipliers, as fields, to be of the same (or higher) perturbative order as the translational and rotational gauge fields. For purely quadratic theories we also make the same assumption of the momenta, since they can be expressed through Eqs. (4.8a) to (4.8d) and Eqs. (4.10a) to (4.10f) linearly in the field perturbations themselves or their gradients and velocities. The latter assumption breaks down in the presence of the Einstein–Hilbert term, since the momentum $\hat{\pi}_{\perp} \sim \mathcal{O}(1)$: as we see in Appendix C.8, it is this property which keeps the sSFCs independent at lowest order. The introduction of multipliers does not restore this feature, since Eq. (5.8) is still *quadratic*.

We will not therefore focus on the new sSFCs or attempt to count the D.o.F on the Minkowski vacuum: for the current theory, this is not the physically interesting background anyway. Instead we turn to the *augmented* PPM, i.e. the PPM including the new multiplier secondaries in Eq. (5.72). Unlike the original SiCs (which are often drafted in at linear order to deal with a vanishing commutator) these new secondaries are surely present in both linear and nonlinear theories. The matrix takes the form



(5.73)

where the specific commutators are provided in Eqs. (C.16a) to (C.16l). We see in the first quadrant of (5.73) an adiaphorous shifting about of the nonlinear commutators from their original positions and values in (5.7). In the second or third quadrants, we see new *linear* commutators which may be used – albeit via a differential equation – to determine the multipliers of all the original PiCs, except for $\varphi_{\perp}$. In general, this is exactly the property we have been seeking.

5.4 Closing remarks

In this final chapter we have extended the original $\text{PGT}^{\text{q},+}$ by introducing geometric multiplier fields which deactivate various parts of the Riemann–Cartan curvature and the torsion. Teleparallel gravity is a special case of this extension. A chronic problem with the original $\text{PGT}^{\text{q},+}$ is the appearance of nonlinear commutators between primary constraints, causing a departure from the linearised constraint structure. This problem is especially severe in purely quadratic theories, where the removal of the Einstein–Hilbert term leads to sparse commutators at linear order. We have examined the effects of geometric multipliers on the canonical analysis. New secondary constraints are produced, even at the *nonlinear* order, which tend not to commute with the original primaries, even at the *linear* order.

While the geometric multipliers therefore fix part of the Hamiltonian problem as it is stated, they are less suitable for patching pre-existing theories. By so changing the linear constraint structure, unwanted modes may be activated and the unitarity destroyed. Such is the case for the purely quadratic theory we developed in Chapters 2 and 3: those same multipliers which do not interfere with the viable cosmology or gravitational waves of the original theory are shown to induce classical ghosts on the flat, torsion-free background.

Crucially, this need not be a problem in the context of our theory, whose viable cosmology attracts towards a background with constant axial torsion. While our theory was first motivated by linear renormalisability and unitarity on the torsionless vacuum, its nonlinear dynamics turn out to select the CS vacuum! It is better to embrace this situation, and we have done so by further pursuing the application of multipliers. In so doing, a clear distinction should first be made between the potential utility of geometric multipliers in future surveys of the old theory space, and a chiefly phenomenological application to our new theory. It is not at all clear that the canonical structure near the CS vacuum will benefit from the use of multipliers. Plausibly, the finite background itelf will already fill the rôle the Einstein–Hilbert term by *inducing* masses. The canonical analysis, along with an assessment of the unitarity and predictivity of the quantum theory, should be addressed in future work.

With this caveat in mind we recall that a previously unresolved aspect of our theory, which contained two remaining arbitrary couplings, was the Newtonian limit. We found that precisely one combination of multipliers is further consistent with this limit on the CS background. These multipliers suppress the 'tensor' parts of the torsion, and are algebraically determined by the 'tensor' part of the spin equation, which allows them to be eliminated in turn from the SET. We refer to this as the 'tensor bypass', since it constitutes a simpler version of the theory with 16 of the 24 torsion components removed and all prior phenomenology preserved. The bypass theory is reached by imposing Eqs. (5.3), (5.25), (5.29), (5.41), (5.49) and (5.52) on Case 2, and we provided its Lagrangian density in Eq. (4). The resulting field equations contain fourth-order terms analogous to Bach and Weyl's conformal gravity, a relic of the traceless (and renormalisable) regime which dominates on the torsionless background. Consequently, one of our two remaining free parameters is identified as the conformal gravity coupling. On the CS background, the fourth-order terms can be treated as high-energy corrections to those of second-order (see e.g. [238]), which themselves express Newton's law of universal gravitation. The Planck mass being built into our theory for a successful cosmology, the final remaining free parameter must then be a natural fraction of the conformal coupling in order to account for falling objects, such as apples.

Closing remarks

This thesis outlines a nascent case for the gravitational gauge theory (2) as a candidate alternative to GR. Gravity is proposed to complement the strong and electroweak forces as a gauge theory of the Poincaré group. Near the Minkowski vacuum, free gravity is then *renormalisable* despite referring explicitly to its own characteristic scales m_{p} and Λ, and *unitary* despite being quadratic in curvature and torsion. Promising though the linear UV theory might seem, it does not extend to the torsionful vacuum selected by the nonlinear IR. But an incongruity between low and high energies is not too surprising: some mechanism must intervene to endow the matter SET with a trace, and indeed it is precisely this VEV which supports the Einsteinian cosmology, emergent dark sector and any accompanying Newtonian limit. These IR features are made all the more remarkable by the *lack* of an Einstein–Hilbert term.

Apart from advocating for the specific theories in Eqs. (2) and (4), we provided general advancements to the field. In Chapter 1 we related the Einstein or Møller energy densities to those of a Klein–Gordon field theory, in a remarkably simple representation of gravitational energetics. In Chapter 3 we built the scalar-tensor analogue of torsion cosmology, revealing for the first time a generally *non-canonical* kinetic structure. In Chapter 5 we developed a new, general theory of multiplier fields, demonstrating their capacity to 'soften' the problematic gauge theory transition from linear to nonlinear dynamics. To help orient future investigation, we conclude with an attempt to summarise from Sections 1.4, 2.6, 3.4, 4.6 and 5.4 the key problems neglected or created by this thesis;

Does the Klein–Gordon correspondence generalise? Its main limitation being static, spherical symmetry, a gravito*magnetic* extension of Chapter 1 requires suitable generalisation of either (i) the isotropic coordinates or (ii) the pseudotensors. Gravitational energy localisation in general [74, 104], and generalisation of Butcher's formalism in particular [90, 87–89], remain open problems.

Are torsion, inflation and the Higgs connected? The dark radiation fraction is connected in Chapter 2 to an initial condition on the pseudoscalar cosmic torsion $Q \equiv \frac{1}{6} n^l \epsilon_{il}{}^{jk} \mathcal{T}^i{}_{jk}$ (for unit timelike n^i), but the primordial circumstances which determine this are unknown. We found in Chapter 5 that unless the 'viable' gravitational QFT near the torsionless vacuum supplies some trace anomaly, it seems confined to conformal states of matter. Such states are abundant in the early Universe, and so we may speculate that the Q VEV vanished only at sufficiently early times prior to the end of the electroweak epoch. What mechanism then sets $Q_{\mathrm{r}} \gtrsim Q_0$ early in the radiation-dominated epoch? In the bypass theory, the torsionless vacuum may be promptly destabilised by ghosts. Such an 'early' conformal symmetry breaking would be especially attractive if it led to torsion-induced inflation [148, 149], with m_{p} facilitating the reheating entropy [239]. Alternatively the process might be 'late', and perhaps even associated with the condensation of the Higgs field.

What is the infrared physics of the new vacuum? The VEV suggests a privileged isotropic torsion frame, so while the Newtonian limit in Chapter 5 is encouraging, extension beyond static sources is not trivial. In this case the theory of *ghost condensates*, whose EFT is known to be stable by a power counting, may serve as an apropos indicator of IR phenomena to expect (e.g. loss of Lorentz invariance) [240–242]. Practically, an understanding of the modified PPN [243–246] and cosmological perturbation [247–249] theories will eventually facilitate far more sophisticated tests [250, 178, 179], perhaps encompassing the Hubble tension as in Chapter 2. We mention that the resulting combined constraints on initial Q_{r} (i.e. $\Delta N_{\mathrm{dr,eff}}$) and coupling $\hat{\alpha}_6$ in (2) (or α_{CG} in (4)) will amount to *direct predictions* of Q_0, which may someday be independently falsifiable [124, 176, 177].

What is the ultraviolet physics of the new vacuum? To further defend our loss of contact with the UV from Chapters 2 to 5, we could appeal to GWS theory: the renormalisable foundations laid by Yang and Mills [251] were long presumed to be spoiled by the phenomena of massive bosons [252]. The dynamical provision of these same masses [253–256], and perturbative renormalisation around the Higgs VEV [257] were not established until much later. Of course this analogy is limited, since the kinetically stalled *Cuscuton* is unrelated to the spontaneously broken $\mathrm{SU}(2)_{\mathrm{L}}$ symmetry: 't Hooft's renormalisation scheme not being expected to apply, we must seek a suitable (perhaps *effective*) alternative.

Why is the cosmological constant *still* so small? As with GR, the torsion self-coupling m_{p} can still be presumed to derive from unknown heavy physics, whereas Λ (which we promote to a similar coupling in Chapter 3) cannot be so easily dismissed. This hierarchy might be better interpreted in the *extended Weyl* counterpart of our theory from Chapter 2, whose dimensionful couplings are interpreted differently [92]. We reiterate that this scale-invariant gauge theory remains profoundly attractive [258], pending a propagator analysis [158]. Separately, we note that a finely-tuned screening of the SM vacuum energy now incurs an additional *unitarity* penalty [153], consistent with the often-assumed 'miraculous' cancellation within L_{M}. Various cancelling-vacuum matter theories are proposed (e.g. unbroken supersymmetry [259]) and none appear wholly satisfying at the current time [260, 30].

Can geometric multipliers fortify the Poincaré gauge theory? Having set out to avoid phenomenological model-building, we view the application in Chapter 5 of multipliers to the viable bypass theory (4) as an IR proof-of-concept, en route to the full theory (2). At the same time, the multipliers were conceived to address erstwhile faults in the nonlinear dynamics of *general* Poincaré gauge theory [168, 169], and the structure of the tensor bypass indicates that they have the *general* capacity to be effective. This intended application is now fit to be investigated as a separate matter.

We emphasise in closing that a classical phenomenology consistent with observation is not lightly to be overlooked in a theory whose prime motivation is quantum mechanical. An apparently functional physical theory may become interesting if, for example, its few free parameters are drawn naturally and without fine tuning from a high-dimensional parameter space. Separately, it may seem compelling when these parameters do not continuously connect to the *paradigmatic model*, here the Ricci scalar. Our theory having demonstrated these various features in the short course of our preliminary investigations, we suggest that it warrants further attention as the potential basis for a more complete description of gravitational interactions.

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Appendix A

Geometric algebra

A.1 Gauging translations

In this appendix we introduce the actively transforming translational gauge field for the geometric algebra formulation of PGT, as used in Section 1.3, and compare with the passive tensor formulation used from Section 2.2 onwards, since this comparison is lacking in the literature. At the same time we will also seek to correct some points in [93]. Familiarity with either Part I of [69] or Chapter 13 of [93] is assumed. The multivector formalism is set in the vector space $\{x\}$ of the STA, rather than a manifold. We use x to denote a particular position vector rather than a set of coordinate functions; the coordinates are written as $\breve{x} \equiv \{x^\mu\}$. Diffeomorphism gauge invariance is interpreted literally, with an active transformation of the fields from their initial value at point x to that at a point $x' \equiv f(x)$

$$\varphi'(x) = \varphi(x'), \tag{A.1}$$

where (A.1) is to be compared to (2.10) and $f(x)$ is an arbitrary (i.e. generally nonlinear) function of x. We can make an equivalent statement which is closer in spirit to (2.10) by defining $x'' \equiv f^{-1}(x)$, so that $\varphi'(x'') = \varphi(x)$, and re-casting the active transformation as a passive one. To this end we define a new set of coordinate functions $\breve{x}' \equiv \{x'^\mu\}$, where $x'^\mu(x) \equiv x^\mu(x'')$. To switch to a coordinate, rather than position based formalism, we use a breve accent to refer to the equivalent function whose arguments are the coordinates. Thus for example $\breve{\varphi}(\breve{x}(x)) = \varphi(x)$ and $\breve{\varphi}(\breve{x}'(x)) = \varphi(f^{-1}(x))$. If we now write all instances of the coordinates assuming them to be evaluated at the original point x rather than x' or x'' (unless otherwise stated), we find $\breve{\varphi}'(\breve{x}') = \breve{\varphi}(\breve{x})$, which has exactly the same meaning as (2.10). Thus the active transformation of the fields in $\{x\}$ comes with a corresponding passive transformation of the fields expressed as coordinate functions at every point x.

We now consider multivector fields, which encode tensors in the geometric algebra. As in Sections 1.2.1 and 2.2.1 we define a basis of vectors $\{\mathsf{e}_\mu\}$ and covectors $\{\mathsf{e}^\mu\}$ as $\mathsf{e}_\mu(x) \equiv \partial x/\partial x^\mu$ and $\mathsf{e}^\mu(x) \equiv \nabla_x x^\mu$, along with the flat-space metric $\mathsf{e}_\mu(x) \cdot \mathsf{e}_\nu(x) \equiv \gamma_{\mu\nu}(x)$, with $\mathsf{e}_\mu(x) \cdot \mathsf{e}^\nu(x) \equiv \delta^\nu_\mu$. The usual passive change of coordinates to the system $\breve{x}' \equiv \{x'^\mu\}$ then invokes precisely the same transformation laws as in (2.11), $\mathsf{e}'^\mu(x) = (\partial x'^\mu/\partial x^\nu)\mathsf{e}^\nu(x)$, $\mathsf{e}'_\mu(x) = (\partial x^\nu/\partial x'^\mu)\mathsf{e}_\nu(x)$ and $\partial'_\mu = (\partial x^\nu/\partial x'^\mu)\partial_\nu$. To establish equivalent quantities which transform actively, we first define $\underline{\mathsf{f}}(a; x) \equiv a \cdot \nabla_x f(x)$, where $\underline{\mathsf{f}}(a; x)$ is a linear function of a and an arbitrary function of x. Returning again to the picture of mapping vector spaces, we note that f induces the outermorphism $\underline{\mathsf{f}}$, which allows us to map vectors and multivectors of arbitrary grade between $\{x\}$ and $\{x'\}$. It is easiest to begin with covectors such as $\{\mathsf{e}^\mu\}$. Noting that $x^\mu = x^\mu(x)$, we find

$$\begin{aligned} a\cdot\mathsf{e}^\mu(x) = a\cdot\nabla_x x^\mu(x) = a\cdot\nabla_x x'^\mu(f(x)) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}(x'^\mu(f(x+\epsilon a)) - x'^\mu(f(x))) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}(x'^\mu(x'+\epsilon\underline{\mathsf{f}}(a;x)) - x'^\mu(x')) = \underline{\mathsf{f}}(a;x)\cdot\nabla_{x'}x'^\mu(x') = a\cdot\bar{\mathsf{f}}(\mathsf{e}'^\mu(x');x). \end{aligned} \tag{A.2}$$

Next, we define the vector-valued displacement gauge field which transforms actively as $\bar{\mathsf{h}}'(a;x) = \bar{\mathsf{h}}(\bar{\mathsf{f}}^{-1}(a;x);x')$, and four *gravity frames* $g^\mu(x) \equiv \bar{\mathsf{h}}(\mathsf{e}^\mu(x);x)$, which transform actively as

$$\bar{\mathsf{h}}'(\mathsf{e}^\mu(x);x) = \bar{\mathsf{h}}(\mathsf{e}'^\mu(x');x') \Rightarrow \bar{\mathsf{h}}'(\mathsf{e}^\mu(x'');x'') = \bar{\mathsf{h}}(\mathsf{e}'^\mu(x);x) \Rightarrow \breve{g}'^\mu(\breve{x}') = \frac{\partial x'^\mu}{\partial x^\nu}\breve{g}^\nu(\breve{x}). \tag{A.3}$$

Thus the gravity frames $\{\breve{g}^\mu(\breve{x})\}$ and basis covectors $\{\breve{\mathsf{e}}^\mu(\breve{x})\}$ obey the same passive transformation law under the GCT corresponding to the active displacement. Previously, $\bar{\mathsf{h}}$ was introduced with the following active motivation [69, 93]. For a covector field $J(x) \equiv \nabla_x\varphi(x)$, we find $\mathcal{J}'(x) \equiv \bar{\mathsf{h}}'(J(x);x) = \bar{\mathsf{h}}(J(x');x') \equiv \mathcal{J}(x')$, and so $\bar{\mathsf{h}}$ is the function we apply to a covector in order to obtain an actively transforming (i.e. 'covariant') quantity. If we re-cast this passively we find the required result $\breve{\mathcal{J}}'(\breve{x}') = \breve{\mathcal{J}}(\breve{x})$. The difference between this and (A.3) stems from the fact that the scalar field φ is taken to be covariant, whereas the scalar fields $\{x^\mu\}$ are not.

Next we will construct actively transforming vectors, beginning with the $\{\mathsf{e}_\mu\}$. Noting that $\mathsf{e}'_\mu(x') = \partial f(x)/\partial x^\mu(x) = \mathsf{e}^\mu(x)\cdot\nabla_x f(x) = \underline{\mathsf{f}}(\mathsf{e}_\mu(x);x)$, we see that the final four gravity frames $g_\mu(x) \equiv \underline{\mathsf{h}}^{-1}(\mathsf{e}_\mu(x);x)$ transform as

$$\underline{\mathsf{h}}'^{-1}(\mathsf{e}_\mu(x);x) = \underline{\mathsf{h}}^{-1}(\mathsf{e}'_\mu(x');x') \Rightarrow \underline{\mathsf{h}}'^{-1}(\mathsf{e}_\mu(x'');x'') = \underline{\mathsf{h}}^{-1}(\mathsf{e}'_\mu(x);x) \Rightarrow \breve{g}'_\mu(\breve{x}') = \frac{\partial x^\nu}{\partial x'^\mu}\breve{g}_\nu(\breve{x}). \tag{A.4}$$

As with the covariantisation of covectors, we need a physical example involving fields to make sense of the function $\underline{\mathsf{h}}^{-1}$ as it acts on vectors. We will use a particle trajectory $x(\lambda)$, where λ is some parameter. Our covector example was the spacetime gradient of a covariant scalar field φ. From the perspective of quantum fields, a particle trajectory is defined by the motion of a wavepacket – a collection of Lehmann–Symanzik–Zimmermann (LSZ) out states – so $x(\lambda)$ represents the final product of a calculation involving, for example, φ, and as such it must already be a covariant quantity. We have to be very careful when understanding *how* the path is covariant however, since [93] states that "Under a displacement the path transforms to $f(x(\lambda))$". We find that the path does the opposite: we have $x'(\lambda) = f^{-1}(x(\lambda))$. We can use this knowledge to calculate the proper time along the path in the presence of a gravitational field, in which the covariant four velocity of the covariant path $x(\lambda)$ and the proper time τ are given by $u(x(\lambda)) \equiv \underline{\mathsf{h}}^{-1}(\partial_\lambda x(\lambda);x(\lambda))$ and $\tau \equiv \int \mathrm{d}\lambda|\underline{\mathsf{h}}^{-1}(\partial_\lambda x(\lambda);x(\lambda))|$. Physically meaningful integrals should be invariant rather than covariant. To check this, we examine the analogous quantity τ' where both the path and the displacement gauge field have been allowed to actively transform

$$\tau' = \int \mathrm{d}\lambda|\underline{\mathsf{h}}'^{-1}(\partial_\lambda f^{-1}(x(\lambda));f^{-1}(x(\lambda)))| = \int \mathrm{d}\lambda|\underline{\mathsf{h}}'^{-1}(\underline{\mathsf{f}}^{-1}(\partial_\lambda x(\lambda);x(\lambda));f^{-1}(x(\lambda)))| = \tau, \tag{A.5}$$

which is all as required. Noting that $\partial_\lambda x(\lambda) = \partial_\lambda x^\mu(x(\lambda))\mathsf{e}_\mu(x(\lambda))$, the passive picture is

$$\tau' = \int \mathrm{d}\lambda \sqrt{|\partial_\lambda x'^\mu(\lambda)\partial_\lambda x'^\nu(\lambda)\breve{g}'_{\mu\nu}(\breve{x}'(\lambda))|} = \int \mathrm{d}\lambda \sqrt{|\partial_\lambda x^\mu(\lambda)\partial_\lambda x^\nu(\lambda)\breve{g}_{\mu\nu}(\breve{x}(\lambda))|} = \tau. \tag{A.6}$$

A.2 Gauging rotations

We will now extend our discussion in Appendix A.1 to the case of Lorentz rotations. A translation from the active rotor formalism into the passive tensor picture of Eq. (2.14) seems to be lacking in the literature, so we now construct one. Consider the multivector field φ which contains arbitrary grades but is guaranteed to transform covariantly under displacements. Under rotations, φ transforms actively according to the rotor transformation law

$$\varphi' = R\varphi\tilde{R}, \tag{A.7}$$

where the rotor is defined as $R \equiv e^{B/2}$, and B is the pure blade encoding both the orientation and magnitude of the rotation. Every grade of the STA constitutes a representation of $\mathrm{SO}^+(1,3)$, however the formula (A.7) does not appear to employ the group generators specific to the grades in φ because details of the generators are absorbed into the geometric product and rotor formalism. To see this, it is helpful to express φ' as an exponential operator acting on φ solely from the LHS. Using the standard derivation in terms of a derivative with respect to a parameter, we apply the Baker–Hausdorff formula for operators to the multivector expression (A.7)

$$\varphi' = \varphi + \frac{1}{2}B\times\varphi + \frac{1}{12}B\times(B\times\varphi) + \dots \tag{A.8}$$

where the usual nested commutators are replaced with the commutator product. Next we define the $\mathrm{SO}^+(1,3)$ generators for the STA $\Sigma_{ij} \equiv \overrightarrow{(\gamma_i\wedge\gamma_j)\times}$, where the arrow signifies that the blade is geometrically commuted with everything that follows to the right[1]. We then find the formula $\varphi' = e^{B^{ij}\Sigma_{ij}/2}\varphi = \underline{\Lambda}(\varphi)$ for the rotation components, where $B^{ij} \equiv (\gamma^i\wedge\gamma^j)\cdot B$ are the rotor components and $\underline{\Lambda}$ is the linear function which performs the Lorentz transformation, with $\Lambda^i{}_j \equiv \gamma^i\cdot\underline{\Lambda}(\gamma_j)$ and $\Lambda_j{}^i \equiv \gamma^i\cdot\bar{\Lambda}(\gamma_j)$. It is worthwhile checking that the generators obey the required Lie algebra $\mathfrak{so}(1,3)$ by calculating the structure constants, and this can be done using the Jacobi identity as it applies to the commutator product

$$[\Sigma_{ij},\Sigma_{kl}]\varphi = ((\gamma_i\wedge\gamma_j)\times(\gamma_k\wedge\gamma_l))\times\varphi = (\eta_{jk}\Sigma_{il} - \eta_{ik}\Sigma_{jl} - \eta_{jl}\Sigma_{ik} + \eta_{il}\Sigma_{jk})\varphi. \tag{A.9}$$

We now want to gauge rotations in accordance with position-dependent blade $B(x)$, so we extend the definition of covariance to $\varphi'(x) = R(x)\varphi(x')\tilde{R}(x) = \underline{\Lambda}(\varphi(x');x')$, or $\varphi'(x'') = R(f^{-1}(x))\varphi(x)\tilde{R}(f^{-1}(x)) = \underline{\Lambda}(\varphi(x);f^{-1}(x))$. Using a notation for the covariant derivative designed to tie in with (2.13), we have

[1] Note that it is convenient to use the *constant* Lorentz basis $\{\gamma_i\}$ and co-basis $\{\gamma^j\}$, where $\gamma_i\cdot\gamma_j \equiv \eta_{ij}$, $\gamma^i\cdot\gamma^j \equiv \eta^{ij}$ and $\gamma_i\cdot\gamma^j \equiv \delta_i^j$. It is perfectly possible to construct a *local* (function of x) Lorentz frame in the STA, if needed, via orthonormal $\{\mathsf{e}_\mu\}$ and $\{\mathsf{e}^\mu\}$. Instead the local Lorentz indices used from Chapter 2 onwards can be assumed in this thesis to translate to the constant STA basis, e.g. we will have $(e_\mu)^i \equiv \gamma^i\cdot\mathsf{e}_\mu$ and $(e^\mu)_i \equiv \gamma_i\cdot\mathsf{e}^\mu$. An advantage of the geometric algebra formulation is that it renders such components unnecessary for formal calculations, since the $\{\gamma_i\}$ and $\{\gamma^i\}$ may be replaced by arbitrary constant vectors, denoted similarly by lower-case Roman letters, e.g. a, b, c, and multivector derivatives with respect to them, ∂_a, ∂_b, ∂_c. These have the desired properties in common with the usual basis and dual basis $\partial_a\cdot a = 4$ and $\partial_a\wedge a = 0$.

$D_\mu\varphi \equiv (g_\mu \cdot \bar{\mathsf{h}}(\nabla) + \omega(g_\mu)\times)\varphi = \partial_\mu\varphi + \Omega(\mathsf{e}_\mu)\times\varphi$, so that $\mathcal{D}_i \equiv \gamma_i \cdot \mathcal{D} \equiv \gamma_i \cdot g^\mu D_\mu$ – we note that there is a slight clash of notation with the covariant derivative acting on spinors, as it appears in [93]. The bivector-valued rotational gauge field is defined as either side of $\omega(\underline{\mathsf{h}}^{-1}(a)) \equiv \Omega(a)$, and using (A.4) we find that it transforms actively on $\{x\}$ or passively as a function of the $\{x^\mu\}$ in the following manner

$$\begin{aligned} \omega'(g'_\mu(x);x) &= R(x)\omega(g'_\mu(x);x')\tilde{R}(x) - 2\partial_\mu R(x)\tilde{R}(x) \\ &= \underline{\Lambda}(\omega(g'_\mu(x);x');x) - \frac{1}{2}(\tilde{R}(x)\gamma^j R(x))\cdot\partial_\mu(\tilde{R}(x)\gamma^i R(x))(\gamma_i\wedge\gamma_j) \\ &= \frac{1}{2}(\gamma^i\wedge\gamma^j)\cdot\omega(g'_\mu(x);x')\underline{\Lambda}(\gamma_j\wedge\gamma_i;x) - \frac{1}{2}\dot{\partial}_\mu\underline{\Lambda}^{-1}(\gamma^j;x)\cdot\dot{\underline{\Lambda}}^{-1}(\gamma^i;x)(\gamma_i\wedge\gamma_j), \\ \omega'(g'{}_\mu(x'');x'') &= \frac{1}{2}(\gamma^i\wedge\gamma^j)\cdot\omega(g'{}_\mu(x'');x)\underline{\Lambda}(\gamma_j\wedge\gamma_i;x'') - \frac{1}{2}\dot{\partial}'{}_\mu\dot{\bar{\Lambda}}(\gamma_i;x'')\wedge\bar{\Lambda}(\gamma^i;x''), \\ \breve{\omega}'(\breve{g}'{}_\mu(\breve{x}');\breve{x}') &= \frac{1}{2}\frac{\partial x^\nu}{\partial x'^\mu}\big((\gamma^i\wedge\gamma^j)\cdot\breve{\omega}(\breve{g}_\mu(\breve{x});\breve{x})\underline{\breve{\Lambda}}(\gamma_j\wedge\gamma_i;\breve{x}') - \dot{\partial}_\mu\dot{\bar{\breve{\Lambda}}}(\gamma_i;\breve{x}')\wedge\bar{\breve{\Lambda}}(\gamma^i;\breve{x}')\big). \end{aligned} \tag{A.10}$$

We then define the components $A^{ij}{}_\mu \equiv (\gamma^j\wedge\gamma^i)\cdot\omega(g_\mu)$ by comparison with (2.14) (although that transformation is a pure rotation)[2].

A.3 Gauge invariance for Gauss' law

In the classical formulation of GR, the geometry of the Riemann space V_4 is intimately connected to the general integral theorem known as Gauss' (or Stokes') theorem, which is expressed using differential forms. Consequently, this theorem is often used to invoke gravitational effects when obtaining physical laws, with the derivation of the Komar mass in Section 1.2.5 being an example. If the same laws arise in gauge theories of gravity, we ask how the integral theorem on flat Minkowski space, M_4, can be of use in obtaining them. This is resolved by insisting on the gauge invariance of all directed integrals. The first example of this principle is the factor of $\det\mathsf{h}^{-1}$ in the action (1.42) which is an integral over the whole of M_4: this factor has an entirely non-trivial effect on the E.o.M.

The next-simplest case is that of a covariant vector $\mathcal{J}$ integrated over an $d=3$ hypersurface ∂V of directed measure d^3x enclosing the $d=4$ volume V. We can think of the integrand as a linear function $\mathsf{L}(a)$ of that measure $\oint_{\partial V}\mathsf{L}(\mathrm{d}^3x) \equiv \oint_{\partial V}\langle\mathcal{J}\underline{\mathsf{h}}^{-1}(\mathrm{d}^3x)I^{-1}\rangle$. The action of the displacement gauge field on the directed measure in this quantity guarantees gauge invariance of the hypersurface integral. Next we apply the fundamental theorem of geometric calculus in M_4 to obtain an integral with directed measure d^4x on V, and expand as follows

$$\begin{aligned} \oint_{\partial V}\mathsf{L}(\mathrm{d}^3x) &= \int_V \dot{\mathsf{L}}(\dot{\nabla}\mathrm{d}^4x) = \int_V\langle\mathcal{J}\underline{\mathsf{h}}^{-1}(\overleftrightarrow{\nabla}\mathrm{d}^4x)I^{-1}\rangle = \int_V\langle\mathcal{J}I\underline{\mathsf{h}}^{-1}(I^{-1}\overleftrightarrow{\nabla})|\mathrm{d}^4x|\rangle \\ &= \int_V|\mathrm{d}^4x|\mathcal{J}\cdot\bar{\mathsf{h}}(\overleftrightarrow{\nabla})\det\mathsf{h}^{-1} = \int_V|\mathrm{d}^4x|\mathcal{D}\cdot\mathcal{J}\det\mathsf{h}^{-1}, \end{aligned} \tag{A.11}$$

where the final equality follows from (1.46). Thus we see how the divergence in the volume naturally inherits the covariant derivative (and by extension, rotational gauge field) from the gauge invariance of the directed measure on the surface.

[2]Note that the indices on $A^{ij}{}_\mu$ are *reversed* in the blade contraction, and this potential source of sign errors is inherited by the Riemann–Cartan tensor in Appendix A.7.

The integral theorem of differential geometry is not confined to hypersurfaces. In particular, we want to consider directed integrals *within* a Cauchy hypersurface Σ_t whose arbitrary geometry depends on its embedding in M_4. This motivates us now to apply the symplectic *vector manifold* theory [100] to gravitation. In Chapter 1 we consider static spacetimes, where it is possible to choose a Σ_t everywhere orthogonal to the Killing field $\mathcal{K}$. To maintain generality, we will instead set up coordinates $\{x^\mu\}$ such that $x^0 = t$, and an orthonormal basis $\gamma_{\mu\nu} = \eta_{\mu\nu}$ with the pseudoscalar on Σ_t defined $\mathring{I} \equiv \mathsf{e}^t \cdot I$, and following the convention of [93] we will write the projection of the vector derivative onto Σ_t as $\mathring{\nabla} \equiv \partial$. The quantities $\mathring{I}$ and $\mathring{\nabla}$ are not gauge covariant, but we will only invoke them as an intermediate step to equate gauge covariant quantities. Since e^t belongs to the cotangent space on M_4, the vector $g^t \equiv \bar{\mathsf{h}}(\mathsf{e}^t)$ is covariant, and always orthogonal to the covariantised tangent vectors in Σ_t, $g_\alpha \equiv \underline{\mathsf{h}}^{-1}(\mathsf{e}_\alpha)$. In the example in Section 1.3.4, we are actually concerned with the integral of a covariant bivector $\mathcal{B}$ over a closed hypersurface ∂V within Σ_t which surrounds some star $\oint_{\partial V} \mathsf{M}(\mathrm{d}^2x) \equiv \oint_{\partial V} \langle \mathcal{B}\underline{\mathsf{h}}^{-1}(\mathrm{d}^2x)I^{-1}\rangle$. If we apply the fundamental theorem of geometric calculus to the embedded integral we have

$$\begin{aligned}\oint_{\partial V} \mathsf{M}(\mathrm{d}^2x) &= \int_V \dot{\mathsf{M}}(\dot{\partial}\mathrm{d}^3x) = \int_V \langle \mathcal{B}\underline{\mathsf{h}}^{-1}((\overleftarrow{\mathring{\nabla}} \cdot \mathring{I}\mathring{I}^{-1})\mathring{I})I^{-1}|\mathrm{d}^3x|\rangle = \int_V \langle \mathcal{B}I\underline{\mathsf{h}}^{-1}(I^{-1}(\overleftrightarrow{\mathring{\nabla}} \wedge \mathsf{e}^t))|\mathrm{d}^3x|\rangle \\ &= \int_V |\mathrm{d}^3x|\bar{\mathsf{h}}(\overleftrightarrow{\mathring{\nabla}}) \cdot (\bar{\mathsf{h}}(\mathsf{e}^t) \cdot \mathcal{B})\det \mathsf{h}^{-1} = \int_V |\mathrm{d}^3x|\mathcal{D} \cdot (g^t \cdot \mathcal{B})\det \mathsf{h}^{-1},\end{aligned} \tag{A.12}$$

where for the second equality we used $\nabla \wedge \mathsf{e}^t = 0$. We can also think of (A.12) as an integral of the gauge-covariant rejection of $\mathcal{D} \cdot \mathcal{B}$ off Σ_t over the bounded region

$$\oint_{\partial V} \langle \mathcal{B}\underline{\mathsf{h}}^{-1}(\mathrm{d}^2x)I^{-1}\rangle = \int_V (\mathcal{D} \cdot \mathcal{B}) \cdot g^t g_t \cdot (\underline{\mathsf{h}}^{-1}(\mathrm{d}^3x)I^{-1}) = \int_V \langle \mathcal{P}_\perp(\mathcal{D} \cdot \mathcal{B})\underline{\mathsf{h}}^{-1}(\mathrm{d}^3x)I^{-1}\rangle. \tag{A.13}$$

Since the gravity frames form a complete set, the identity operator may be defined as $g^\mu \cdot (\mathcal{D} \cdot \mathcal{B})g_\mu = \mathcal{D} \cdot \mathcal{B}$, so $\mathcal{P}_\perp(\mathcal{D} \cdot \mathcal{B})$ is the only part which survives inside the scalar part of the integrand in (A.13), giving

$$\oint_{\partial V} \langle \mathcal{B}\underline{\mathsf{h}}^{-1}(\mathrm{d}^2x)I^{-1}\rangle = \int_V \langle \mathcal{D} \cdot \mathcal{B}\underline{\mathsf{h}}^{-1}(\mathrm{d}^3x)I^{-1}\rangle, \tag{A.14}$$

which is the required result. Note that (A.11) may also be written in this 'scalar part' form.

A.4 Unitary form of Møller's pseudotensor

We would like to verify that the quantity in (1.63) derived through variational principles is indeed the same as that arrived at in [94] using the unitary form. In our frame-free notation, this form may be written as a sum of two terms, with the second term containing a single displacement gauge field gradient

$$\begin{aligned}\kappa_{\mathrm{M}}\bar{\mathsf{t}}_{\mathrm{G}}(n)\det \mathsf{h} = \partial_c\bar{\mathsf{h}}(n \wedge \partial_a \wedge \partial_b) \cdot \Big[\frac{1}{2}(\Omega(a) \times \Omega(b)) \wedge \underline{\mathsf{h}}^{-1}(c) \\ + \Omega(b) \wedge (\Omega(a) \cdot \underline{\mathsf{h}}^{-1}(c))\Big] + \partial_c\bar{\mathsf{h}}(n \wedge \partial_a \wedge \partial_b) \cdot (\Omega(b) \wedge \underline{\mathsf{h}}^{-1}(c)_{,a}).\end{aligned} \tag{A.15}$$

To prove the equivalence, we start by expanding this second term

$$\begin{aligned}\partial_c\bar{\mathsf{h}}(n\wedge\partial_a\wedge\partial_b)\cdot(\Omega(b)\wedge\underline{\mathsf{h}}^{-1}(c)_{,a}) &= \dot{\bar{\mathsf{h}}}^{-1}(\omega(b)\cdot(\bar{\mathsf{h}}(\dot{\nabla})\wedge\partial_b\wedge\bar{\mathsf{h}}(n)))\\ &= \bar{\mathsf{h}}^{-1}\dot{\bar{\mathsf{h}}}(\dot{\nabla})(\bar{\mathsf{h}}(n)\wedge\partial_b)\cdot\omega(b) - \dot{\nabla}(\dot{\bar{\mathsf{h}}}(n)\wedge\partial_b)\cdot\omega(b)\\ &\quad - \dot{\nabla}(\bar{\mathsf{h}}(n)\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(\partial_b))\cdot\omega(b) + \bar{\mathsf{h}}^{-1}[(\bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}(n))\cdot(\partial_b\cdot\omega(b))\\ &\quad - (\bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(\partial_b))\cdot(\bar{\mathsf{h}}(n)\cdot\omega(b))].\end{aligned} \tag{A.16}$$

By substituting for various displacement gauge field gradients using (1.45) and (1.46) we find

$$\begin{aligned}\partial_c\bar{\mathsf{h}}(n\wedge\partial_a\wedge\partial_b)\cdot(\Omega(b)\wedge\underline{\mathsf{h}}^{-1}(c)_{,a}) &= \kappa({}_{\mathrm{M}}\mathfrak{t}_{\mathrm{G}}(n)+n_{\mathrm{M}}\mathscr{L}_{\mathrm{G}})\det\mathsf{h} + \bar{\mathsf{h}}^{-1}[(\bar{\mathsf{h}}(n)\cdot\omega(b))\cdot(\partial_c\wedge(\partial_b\cdot\omega(b)))\\ &\quad - (\partial_b\cdot\omega(b))\cdot(\partial_c\wedge(\bar{\mathsf{h}}(n)\cdot\omega(b))) + (\bar{\mathsf{h}}(n)\wedge\partial_b)\cdot\omega(b)(\partial_c\cdot\omega(c))],\end{aligned} \tag{A.17}$$

thus the second term in (A.15) collects the three 'kinetic' terms in (1.63), leaving a remainder which may be expressed purely in terms of the $\omega(a)$ fields. If we now turn to the first term in (A.15), which is identified as *Møller's superpotential*, we have

$$\begin{aligned}\partial_c\bar{\mathsf{h}}(n\wedge\partial_a\wedge\partial_b)\cdot\left[\frac{1}{2}(\Omega(a)\times\Omega(b))\wedge\underline{\mathsf{h}}^{-1}(c)+\Omega(b)\wedge(\Omega(a)\cdot\underline{\mathsf{h}}^{-1}(c))\right] &= -\kappa n_{\mathrm{M}}\mathscr{L}_{\mathrm{G}}\det\mathsf{h}\\ - \frac{1}{2}(\bar{\mathsf{h}}(n)\cdot(\omega(a)\times\omega(b)))\cdot(\partial_a\wedge\partial_b) - \omega(a)\cdot[\omega(b)\cdot(\bar{\mathsf{h}}(n)\wedge\partial_a\wedge\partial_b)].\end{aligned} \tag{A.18}$$

After expanding, the trailing terms in (A.17) and (A.18) cancel exactly.

A.5 Killing fields and the first Bianchi identity

In Section 1.3.1 we mention the 'double wedge' equation $\mathcal{D}\wedge(\mathcal{D}\wedge M)=0$, which is true for *any* multivector M. By contracting with an arbitrary constant basis trivector and expanding the resultant scalar, one can see that this is a statement of the first Bianchi identity, and hence a symmetry property of the Riemann–Cartan tensor. Rather than show this here, we will use the same approach to obtain a very useful result regarding the second covariant derivative of a Killing field, $M=\mathcal{K}$. Hence we write

$$\begin{aligned}(a\wedge b\wedge c)\cdot(\mathcal{D}\wedge(\mathcal{D}\wedge\mathcal{K}) &= a\cdot(b\cdot((c\cdot\overrightarrow{\mathcal{D}})\mathcal{D})\mathcal{K}) - b\cdot(a\cdot((c\cdot\overrightarrow{\mathcal{D}})\mathcal{D})\mathcal{K})\\ -a\cdot(c\cdot((b\cdot\overrightarrow{\mathcal{D}})\mathcal{D})\mathcal{K}) + b\cdot(c\cdot((a\cdot\overrightarrow{\mathcal{D}})\mathcal{D})\mathcal{K}) &+ c\cdot(a\cdot((b\cdot\overrightarrow{\mathcal{D}})\mathcal{D})\mathcal{K}) - c\cdot(b\cdot((a\cdot\overrightarrow{\mathcal{D}})\mathcal{D})\mathcal{K}).\end{aligned} \tag{A.19}$$

By applying the Killing equation to certain terms in this expansion we arrive at $a\cdot(b\cdot((c\cdot\overrightarrow{\mathcal{D}})\mathcal{D})\mathcal{K}) = -c\cdot((a\wedge b)\cdot(\overrightarrow{\mathcal{D}}\wedge\mathcal{D})\mathcal{K}) = c\cdot(\mathcal{R}(a\wedge b)\cdot\mathcal{K})$. In particular, if we set $c=\partial_b$ we find the required result

$$a\cdot(\overrightarrow{\mathcal{D}}\cdot\mathcal{D}\mathcal{K}) = \mathcal{R}(a)\cdot\mathcal{K}. \tag{A.20}$$

A.6 Derivatives and the metric determinant

In GTG we have from [69, 93] for the derivative of the metric determinant $\partial_{\bar{\mathsf{h}}(a)}\det\mathsf{h}^{-1} = -\det\mathsf{h}^{-1}\underline{\mathsf{h}}^{-1}(a)$. We note in passing that the geometric algebra determinant is equivalent to the quantities $b\equiv\det(b^i{}_\mu)\equiv h^{-1}\equiv 1/\det(h_i{}^\mu)\equiv\det\mathsf{h}^{-1}$ from Section 2.2.2. When derivatives are present, we can invoke an orthonormal frame to give

$$\partial_{\partial_\mu \bar{\mathsf{h}}(c)} \partial_\nu \det \mathsf{h}^{-1} = -\partial_{\partial_\mu \bar{\mathsf{h}}(c)} (\partial_\nu \underline{\mathsf{h}}^{-1}(\gamma_0) \wedge \cdots \wedge \underline{\mathsf{h}}^{-1}(\gamma_3) + \underline{\mathsf{h}}^{-1}(\gamma_0) \wedge \partial_\nu \underline{\mathsf{h}}^{-1}(\gamma_1) \wedge \cdots \wedge \underline{\mathsf{h}}^{-1}(\gamma_3) + \dots) I. \tag{A.21}$$

On the other hand, the Leinbiz rule produces a very similar expansion in the original expression

$$\partial_{\bar{\mathsf{h}}(c)} \det \mathsf{h}^{-1} = -\dot{\partial}_{\bar{\mathsf{h}}(c)} (\dot{\underline{\mathsf{h}}}^{-1}(\gamma_0) \wedge \cdots \wedge \underline{\mathsf{h}}^{-1}(\gamma_3) + \underline{\mathsf{h}}^{-1}(\gamma_0) \wedge \dot{\underline{\mathsf{h}}}^{-1}(\gamma_1) \wedge \cdots \wedge \underline{\mathsf{h}}^{-1}(\gamma_3) + \dots) I, \tag{A.22}$$

so by comparison the two must be the same up to δ^μ_ν, or $\partial_{\bar{\mathsf{h}}(c)_{,n}} (\det \mathsf{h}^{-1})_{,b} = -(n \cdot b) \det \mathsf{h}^{-1} \underline{\mathsf{h}}^{-1}(c)$.

A.7 Quadratic invariants

In this appendix we will construct the field strength tensors of PGT, and the 'geometric algebra form' of the quadratic action which motivates the coupling parameters used throughout Sections 2.5 to 2.6. The Riemann–Cartan curvature and torsion tensors are represented by bivector-valued linear functions their bivector and vector arguments, with the usual components recovered as scalars via the appropriate interior product[3]

$$\mathcal{R}_{ijkl} \equiv (\gamma_j \wedge \gamma_i) \cdot \mathcal{R}(\gamma_k \wedge \gamma_l), \quad \mathcal{R}(a \wedge b) \equiv a \cdot \bar{\mathsf{h}}(\nabla) \omega(b) - b \cdot \bar{\mathsf{h}}(\nabla) \omega(a) + \omega(a) \times \omega(b), \tag{A.23a}$$

$$\mathcal{T}^i_{\ jk} \equiv (\gamma_j \wedge \gamma_k) \cdot \mathcal{T}(\gamma^i), \quad \mathcal{T}(a) \equiv \bar{\mathsf{h}}(\partial_b) \wedge (b \cdot \dot{\nabla} \dot{\bar{\mathsf{h}}}(\bar{\mathsf{h}}^{-1}(a)) + \Omega(b) \cdot a). \tag{A.23b}$$

The definitions Eqs. (A.23a) and (A.23b) can be compared component-wise with Eqs. (2.16) and (2.17), respectively. Then we can define the vector-valued Ricci tensor, torsion contraction and Ricci scalar $\mathcal{R}(a) \equiv \partial_b \cdot \mathcal{R}(b \wedge a)$, $\mathcal{R} \equiv \partial_a \cdot \mathcal{R}(a)$ and $\mathcal{T} \equiv \partial_a \cdot \mathcal{T}(a)$, with components $\mathcal{R}^i{}_k \equiv \gamma^i \cdot \mathcal{R}(\gamma_k)$ and $\mathcal{T}_j \equiv \gamma_j \cdot \mathcal{T}$. The essential symmetries $\mathcal{R}_{(ij)kl} \equiv \mathcal{R}_{ij(kl)} \equiv \mathcal{T}^i_{\ (jk)} \equiv 0$ follow immediately from Eqs. (A.23a) and (A.23b). Less obvious are those symmetries of the Riemann–Cartan and Ricci tensors which emerge in the metrical limit of vanishing torsion. To discuss these, we define the *adjoint* functions $(a \wedge b) \cdot \mathcal{R}(c \wedge d) \equiv \bar{\mathcal{R}}(a \wedge b) \cdot (c \wedge d)$ and $a \cdot \mathcal{R}(b) \equiv \bar{\mathcal{R}}(a) \cdot b$, which are distinguishable from the functions themselves only when torsion is present. Without torsion, the overbars can be removed and by inserting the Lorentz basis we recover $\mathcal{R}_{ijkl} \equiv \mathcal{R}_{klij}$ and $\mathcal{R}_{[ij]} \equiv 0$.

A natural reshuffling of the gravitational action is possible within the STA. The usual arrangement of quadratic invariants such as (2.24) and (2.25) are obtained by asking for all unique contraction permutations between squared tensors. Alternatively, we can ask for all unique geometric quantities formed from the same tensor, and square them. Applied to the quadratic Riemann–Cartan sector, most of the terms in either decomposition are identical, for example $\mathcal{R}_{ijkl}\mathcal{R}^{ijkl} = 2\mathcal{R}(c \wedge d) \cdot \mathcal{R}(\partial_d \wedge \partial_c)$ and $\mathcal{R}_{ijkl}\mathcal{R}^{klij} = 2\bar{\mathcal{R}}(c \wedge d) \cdot \mathcal{R}(\partial_d \wedge \partial_c)$, with analogous formulae in the quadratic Ricci sector. The only theory parameter that requires extra care in its conversion is α_5. Tellingly this is the only quadratic invariant that is not generated by a clean symmetry operation on its Riemann–Cartan tensor factors: $\mathcal{R}_{ijkl}\mathcal{R}^{ikjl} = ((b \cdot \bar{\mathcal{R}}(d \wedge c)) \cdot (\partial_c \cdot \mathcal{R}(\partial_d \wedge \partial_b))$. This quantity does not conform to the principle of the new decomposition, but can itself be further decomposed using $(\partial_b \wedge \mathcal{R}(b \wedge d)) \cdot (c \wedge \mathcal{R}(\partial_c \wedge \partial_d)) = (c \cdot \mathcal{R}(b \wedge d)) \cdot (\partial_b \cdot \mathcal{R}(\partial_c \wedge \partial_d)) - \mathcal{R}(d \wedge c) \cdot \mathcal{R}(\partial_c \wedge \partial_d)$. This results in the following decomposition of

[3] Note the unfortunate reversal in the first two indices of the Riemann–Cartan tensor, inherited from the the rotational gauge field in Appendix A.2.

the quadratic Riemann–Cartan sector

$$\begin{aligned} L_{\mathcal{R}^2} &= \breve{\alpha}_1\mathcal{R}^2 + \breve{\alpha}_2\mathcal{R}(\partial_b)\cdot\mathcal{R}(b) + \breve{\alpha}_3\bar{\mathcal{R}}(\partial_b)\cdot\mathcal{R}(b) + \breve{\alpha}_4\mathcal{R}(\partial_b\wedge\partial_c)\cdot\mathcal{R}(c\wedge b) \\ &\quad + \breve{\alpha}_5\left(\partial_b\wedge\mathcal{R}\left(b\wedge d\right)\right)\cdot\left(c\wedge\mathcal{R}\left(\partial_c\wedge\partial_d\right)\right) + \breve{\alpha}_6\bar{\mathcal{R}}(\partial_b\wedge\partial_c)\cdot\mathcal{R}(c\wedge b), \end{aligned} \tag{A.24}$$

while the same methodology decomposes the quadratic torsion sector as follows

$$L_{\mathcal{T}^2} = \breve{\beta}_1\mathcal{T}(\partial_b)\cdot\mathcal{T}(b) + \breve{\beta}_2\left(\partial_a\wedge\mathcal{T}(a)\right)\cdot\left(\partial_b\wedge\mathcal{T}(b)\right) + \breve{\beta}_3\mathcal{T}^2. \tag{A.25}$$

The decompositions in (A.24) and (A.25) are the origin of the theory parameters in Eqs. (B.23d) to (B.23f). Note that the fifth term on the RHS of (A.24) and the second term on the RHS of (A.25) are the squares of the Riemann–Cartan and torsion *protractions* which were mentioned in Section 2.4.2.

A.8 Conformal gravity vs k-screened gravity

In this appendix we prove the claim that was made in Section 2.6, that the k-screening mechanism does not need to follow from a torsionful generalisation of CG. The Weyl–Cartan tensor is defined as $\mathcal{W}(a\wedge b)\equiv\mathcal{R}(a\wedge b)-\mathcal{S}(a)\owedge b$, where the Schouten tensor is defined in terms of the Ricci tensor and scalar as $\mathcal{S}(a)\equiv\frac{1}{2}(\mathcal{R}(a)-\frac{1}{6}\mathcal{R}a)$, and in geometric algebra the Kulkarni–Nomizu product of two tensors (represented as usual by linear functions on vectors) is $\mathcal{A}(a)\owedge\mathcal{B}(b)\equiv\mathcal{A}(a)\wedge\mathcal{B}(b)-\mathcal{B}(a)\wedge\mathcal{A}(b)$. This allows us to translate the Weyl–Cartan tensor directly into the Riemann–Cartan and Ricci as follows

$$\mathcal{W}(a\wedge b)\equiv\mathcal{R}(a\wedge b)-\frac{1}{2}\left(\mathcal{R}(a)\wedge b+a\wedge\mathcal{R}(b)\right)+\frac{1}{6}a\wedge b\mathcal{R}. \tag{A.26}$$

The adjoint Weyl–Cartan tensor in the presence of torsion can then be obtained as $\bar{\mathcal{W}}(a\wedge b)\equiv\bar{\mathcal{R}}(a\wedge b)-\frac{1}{2}\left(\bar{\mathcal{R}}(a)\wedge b+a\wedge\bar{\mathcal{R}}(b)\right)+\frac{1}{6}a\wedge b\mathcal{R}$. While is not possible by invoking torsion to resurrect the contractions of the Weyl–Cartan tensor or its adjoint $\partial_a\cdot\mathcal{W}(a\wedge b)\equiv\partial_a\cdot\bar{\mathcal{W}}(a\wedge b)\equiv 0$, we do find that the Weyl–Cartan protraction no longer vanishes in general $\partial_a\wedge\mathcal{W}(a\wedge b)\equiv\partial_a\wedge\mathcal{R}(a\wedge b)-\frac{1}{2}\partial_a\wedge\mathcal{R}(a)\wedge b$.

By combining these results and by analogy with the six quadratic curvature invariants, we find three obvious candidates for the quadratic invariants of the Weyl–Cartan

$$\begin{aligned} \left(\partial_a\wedge\mathcal{W}(a\wedge b)\right)\cdot\left(c\wedge\mathcal{W}(\partial_c\wedge\partial_b)\right) &= \left(\partial_a\wedge\mathcal{R}(a\wedge b)\right)\cdot\left(c\wedge\mathcal{R}(\partial_c\wedge\partial_b)\right)+\frac{1}{2}\left(\mathcal{R}(\partial_a)-\bar{\mathcal{R}}(\partial_a)\right)\cdot\mathcal{R}(a), \\ \mathcal{W}(\partial_b\wedge\partial_a)\cdot\mathcal{W}(a\wedge b) &= \mathcal{R}(\partial_b\wedge\partial_a)\cdot\mathcal{R}(a\wedge b)-\mathcal{R}(\partial_a)\cdot\mathcal{R}(a)+\frac{1}{6}\mathcal{R}^2, \\ \bar{\mathcal{W}}(\partial_b\wedge\partial_a)\cdot\mathcal{W}(a\wedge b) &= \bar{\mathcal{R}}(\partial_b\wedge\partial_a)\cdot\mathcal{R}(a\wedge b)-\bar{\mathcal{R}}(\partial_a)\cdot\mathcal{R}(a)+\frac{1}{6}\mathcal{R}^2. \end{aligned} \tag{A.27}$$

This motivates the three further theory parameters $\breve{\mu}_1\equiv\frac{1}{6}\breve{\alpha}_1-\breve{\alpha}_2+\breve{\alpha}_4$, $\breve{\mu}_2\equiv\frac{1}{6}\breve{\alpha}_1-\breve{\alpha}_3+\breve{\alpha}_6$ and $\breve{\mu}_3\equiv\frac{1}{2}\breve{\alpha}_2-\frac{1}{2}\breve{\alpha}_3+\breve{\alpha}_5$ for the quadratic Weyl–Cartan sector. It is then clear that the k-screening condition (2.53) is indeed compatible with *any* generalisation of the CG theory to nonzero torsion, since $\breve{\boldsymbol{\mu}}_1\cdot\boldsymbol{\sigma}_3=\breve{\boldsymbol{\mu}}_2\cdot\boldsymbol{\sigma}_3=\breve{\boldsymbol{\mu}}_3\cdot\boldsymbol{\sigma}_3=0$. Moreover, we may relate some of the more specific cosmologies (e.g. Class ^{4}H defined by (B.6)) mentioned in Section 2.6 to the quadratic Weyl–Cartan sector, since $\breve{\boldsymbol{\mu}}_1\cdot\boldsymbol{\sigma}_1=\breve{\boldsymbol{\mu}}_2\cdot\boldsymbol{\sigma}_2=0$.

However, the parameter space of the quadratic Weyl–Cartan sector is *three* dimensional, whilst that of the quadratic Riemann–Cartan sector is *five* dimensional as discussed in Section 2.5.2. It should therefore be possible to construct a fourth theory which is simultaneously k-screened and independent of the quadratic Weyl–Cartan sector.

A.9 The gravitational field equations

Having provided the the geometric algebra formulation of standard $\mathrm{PGT}^{\mathrm{q},+}$ in Appendices A.1, A.2 and A.7, we will now construct the field equations for the general extension to geometric multipliers developed in Chapter 5. The gravitational Lagrangian (5.8) takes the form

$$\begin{aligned}L_{\mathrm{G}} &= \big[\check{\alpha}_1\mathcal{R} + \acute{\alpha}_1\lambda_{\mathcal{R}}\big]\mathcal{R} + \big[\check{\alpha}_2\mathcal{R}(\partial_a) + \acute{\alpha}_2\lambda_{\mathcal{R}}(\partial_a)\big]\cdot\mathcal{R}(a) + \big[\check{\alpha}_3\bar{\mathcal{R}}(\partial_a) + \acute{\alpha}_3\bar{\lambda}_{\mathcal{R}}(\partial_a)\big]\cdot\mathcal{R}(a)\\ &+ \big[\check{\alpha}_4\mathcal{R}(\partial_a\wedge\partial_b) + \acute{\alpha}_4\lambda_{\mathcal{R}}(\partial_a\wedge\partial_b)\big]\cdot\mathcal{R}(b\wedge a) + \partial_b\wedge\big[\check{\alpha}_5\mathcal{R}(b\wedge d) + \acute{\alpha}_5\lambda_{\mathcal{R}}(b\wedge d)\big]\cdot c\wedge\mathcal{R}(\partial_c\wedge\partial_d)\\ &+ \big[\check{\alpha}_6\bar{\mathcal{R}}(\partial_a\wedge\partial_b) + \acute{\alpha}_6\bar{\lambda}_{\mathcal{R}}(\partial_a\wedge\partial_b)\big]\cdot\mathcal{R}(b\wedge a) + m_{\mathrm{p}}{}^2\big[\check{\beta}_1\mathcal{T}(\partial_a) + \acute{\beta}_1\lambda_{\mathcal{T}}(\partial_a)\big]\cdot\mathcal{T}(a)\\ &+ m_{\mathrm{p}}{}^2\partial_a\wedge\big[\check{\beta}_2\mathcal{T}(a) + \acute{\beta}_2\lambda_{\mathcal{T}}(a)\big]\cdot\partial_b\wedge\mathcal{T}(b) + m_{\mathrm{p}}{}^2\big[\check{\beta}_3\mathcal{T} + \acute{\beta}_3\lambda_{\mathcal{T}}\big]\cdot\mathcal{T},\end{aligned}\tag{A.28}$$

where $\lambda_{\mathcal{R}}(a\wedge b)$ and $\lambda_{\mathcal{T}}(a)$ are bivector-valued multipliers which share their contraction notation (Ricci and torsion contractions from Appendix A.7) with the respective field strength tensors, $\lambda^{ij}{}_{kl} \equiv (\gamma^j\wedge\gamma^i)\cdot\lambda_{\mathcal{R}}(\gamma_k\wedge\gamma_l)$ and $\lambda^i{}_{jk} \equiv (\gamma_j\wedge\gamma_k)\cdot\lambda_{\mathcal{T}}(\gamma^i)$. Unlike in Chapter 5, we need extra labels to distinguish here between the torsion and Riemann–Cartan multipliers because in geometric algebra a linear function distributes over the wedge product. For translation of the couplings in (A.28), which are derived from Appendix A.7, the reader is directed to Eqs. (B.23d) to (B.23f), with identical relations between e.g. the $\{\acute{\alpha}_I\}$ and $\{\bar{\alpha}_I\}$, and the $\{\acute{\beta}_M\}$ and $\{\bar{\beta}_M\}$.

In order to construct the field equations, we define the derivative with respect to a bivector-valued linear function of a bivector argument[4] as $\partial_{\underline{\mathsf{f}}(c\wedge d)}\langle\underline{\mathsf{f}}(a\wedge b)\mathcal{B}\rangle \equiv (c\wedge d)\cdot(a\wedge b)\mathcal{B}$, for constant bivector $\mathcal{B}$. The generalised momenta in Eqs. (5.13a) and (5.13b) are now defined as $\pi_{\mathcal{T}}(a) \equiv \partial_{\mathcal{T}(a)}\det\mathsf{h}^{-1}L_{\mathrm{G}}$ and $\pi_{\mathcal{R}}(a\wedge b) \equiv -\partial_{\mathcal{R}(a\wedge b)}\det\mathsf{h}^{-1}L_{\mathrm{G}}$, with components [5] $\pi_{ij}{}^{kl} \equiv (\gamma_j\wedge\gamma_i)\cdot\pi_{\mathcal{R}}(\gamma^k\wedge\gamma^l)$ and $\pi_i{}^{jk} \equiv (\gamma^j\wedge\gamma^k)\cdot\pi_{\mathcal{T}}(\gamma_i)$. From this point, we then obtain the actual momenta for the full theory (A.28)

$$\begin{aligned}\pi_{\mathcal{T}}(a) = -m_{\mathrm{p}}{}^2\det\mathsf{h}^{-1}\Big[&\big[2\check{\beta}_1\mathcal{T}(a) + \acute{\beta}_1\lambda_{\mathcal{T}}(a)\big] + a\cdot\partial_b\wedge\big[2\check{\beta}_2\mathcal{T}(b) + \acute{\beta}_2\lambda_{\mathcal{T}}(b)\big]\\ &- a\wedge\Big(\partial_b\cdot\big[2\check{\beta}_3\mathcal{T}(b) + \acute{\beta}_3\lambda_{\mathcal{T}}(b)\big]\Big)\Big],\end{aligned}\tag{A.29a}$$

$$\begin{aligned}\pi_{\mathcal{R}}(a\wedge b) = -\det\mathsf{h}^{-1}\Big[&2\big[2\check{\alpha}_1\mathcal{R} + \acute{\alpha}_1\lambda_{\mathcal{R}}\big](a\wedge b) + \Big(2\check{\alpha}_2\big[\mathcal{R}(a)\wedge b + a\wedge\mathcal{R}(b)\big]\\ &+\acute{\alpha}_2\big[\lambda_{\mathcal{R}}(a)\wedge b + a\wedge\lambda_{\mathcal{R}}(b)\big]\Big) + \Big(2\check{\alpha}_3\big[\bar{\mathcal{R}}(a)\wedge b + a\wedge\bar{\mathcal{R}}(b)\big] + \acute{\alpha}_3\big[\bar{\lambda}_{\mathcal{R}}(a)\wedge b + a\wedge\bar{\lambda}_{\mathcal{R}}(b)\big]\Big)\\ &+2\big[2\check{\alpha}_4\mathcal{R}(a\wedge b) + \acute{\alpha}_4\lambda_{\mathcal{R}}(a\wedge b)\big] + \Big(2\check{\alpha}_5\big[\partial_c\wedge\mathcal{R}(c\wedge a)\cdot b - a\cdot\partial_c\wedge\mathcal{R}(c\wedge b)\big]\\ &+\acute{\alpha}_5\big[\partial_c\wedge\lambda_{\mathcal{R}}(c\wedge a)\cdot b - a\cdot\partial_c\wedge\lambda_{\mathcal{R}}(c\wedge b)\big]\Big) + 2\big[2\check{\alpha}_6\bar{\mathcal{R}}(a\wedge b) + \acute{\alpha}_6\bar{\lambda}_{\mathcal{R}}(a\wedge b)\big]\Big].\end{aligned}\tag{A.29b}$$

[4] We obtain this from $\partial_{f^{ij}{}_{kl}}\langle\underline{\mathsf{f}}(a\wedge b)\mathcal{B}\rangle = (\gamma^l\wedge\gamma^k)\cdot(a\wedge b)(\gamma_i\wedge\gamma_j)\cdot\mathcal{B}$, where the tensor components are as with the Riemann–Cartan curvature $f^{ij}{}_{kl} \equiv (\gamma^j\wedge\gamma^i)\cdot\underline{\mathsf{f}}(\gamma_k\wedge\gamma_l)$, and comparing with the identity $\underline{\mathsf{f}}(c\wedge d) = \frac{1}{4}(\gamma_i\wedge\gamma_j)(c\wedge d)\cdot(\gamma^l\wedge\gamma^k)f^{ij}{}_{kl}$.

[5] In order to recover the components it is important to observe that we are differentiating with respect to *bivectors*. For bivectors $\mathcal{B}$ and $\mathcal{C}$ we have $(\gamma_i\wedge\gamma_j)\cdot\big(\partial_{\mathcal{C}}\langle\mathcal{C}\mathcal{B}\rangle\big) = \mathcal{B}_{ij}$ and $\partial_{\mathcal{C}^{ij}}\langle\mathcal{C}\mathcal{B}\rangle = \mathcal{B}_{ji}$, where the (usual) convention for labelling the indices of a bivector is that $\mathcal{B}_{ij} \equiv (\gamma_i\wedge\gamma_j)\cdot\mathcal{B}$.

We next define the stress-energy vector density of the matter sector as

$$\underline{\tau}(a) \equiv -\delta_{\underline{\mathsf{h}}^{-1}(a)} \int |\mathrm{d}^4 x| \det \mathsf{h}^{-1} L_{\mathrm{M}}, \quad \tau^{\mu}{}_{i} \equiv \gamma_i \cdot \underline{\tau}(\mathsf{e}^{\mu}). \tag{A.30}$$

The 'mixed-index' field strength tensors are given by $\mathcal{R}(a \wedge b) \equiv \mathsf{R}(\underline{\mathsf{h}}(a \wedge b))$ and $\mathcal{T}(a) \equiv \bar{\mathsf{h}}(\mathsf{T}(a))$, where the first equality is defined already in [93, 69] and we have $R^{ij}{}_{\mu\nu} \equiv (\gamma^j \wedge \gamma^i) \cdot \mathsf{R}(\mathsf{e}_\mu \wedge \mathsf{e}_\nu)$ and $T^{i}{}_{\mu\nu} \equiv (\mathsf{e}_\mu \wedge \mathsf{e}_\nu) \cdot \mathsf{T}(\gamma^i)$. Now we can see that the Riemann–Cartan tensor in the geometric algebra formulation is actually defined 'inside out' with respect to the torsion. Proceeding in the spirit of our previous arguments we find after some long calculations that $\partial_{\underline{\mathsf{h}}^{-1}(q)} \langle \mathcal{R}(a \wedge b) \mathcal{B} \rangle = \frac{1}{2} \left[\bar{\mathsf{h}}(q) \cdot (a \wedge b) \right] \cdot (\partial_f \wedge \partial_e) \langle \mathcal{R}(e \wedge f) \mathcal{B} \rangle$ and $\bar{\partial}_{\underline{\mathsf{h}}^{-1}(q)} \langle \mathcal{T}(a) \mathcal{B} \rangle = \big(\bar{\mathsf{h}}(q) \cdot \mathcal{B} \big) \cdot \mathcal{T}(a)$, where the overbar on the derivative is introduced in (C.33), and indicates here that the mixed-index $\mathsf{R}(a \wedge b)$ and $\mathsf{T}(a)$ be held constant. From here we can see that

$$\bar{\partial}_{\underline{\mathsf{h}}^{-1}(q)} \det \mathsf{h}^{-1} L_{\mathrm{G}} = \mathcal{R}(\partial_a \wedge \partial_b) \cdot \pi_{\mathcal{R}}(b \wedge \bar{\mathsf{h}}(q)) a + \mathcal{T}(\partial_a) \cdot \big(\bar{\mathsf{h}}(q) \cdot \pi_{\mathcal{T}}(a) \big) + \det \mathsf{h}^{-1} L_{\mathrm{G}} \bar{\mathsf{h}}(q). \tag{A.31}$$

For the surface part of the variations, we notice that $\partial_{\underline{\mathsf{h}}^{-1}(q),p} \langle \mathcal{T}(a) \mathcal{B} \rangle = -\bar{\mathsf{h}}(p \wedge q) \cdot \mathcal{B} a$, and combining this with (A.31) and (A.30), and keeping track of the connection, we find the stress-energy equation to be

$$\begin{aligned} \underline{\tau}(a) = & -\bar{\mathsf{h}}(a \wedge \overleftrightarrow{D}) \cdot \pi_{\mathcal{T}}(\partial_b) b + \mathcal{T}(\partial_b) \cdot (\pi_{\mathcal{T}}(b) \cdot \bar{\mathsf{h}}(a)) + \mathcal{R}(\partial_b \wedge \partial_c) \cdot \pi_{\mathcal{R}}(c \wedge \bar{\mathsf{h}}(a)) b \\ & + \det \mathsf{h}^{-1} L_{\mathrm{G}} \bar{\mathsf{h}}(a). \end{aligned} \tag{A.32}$$

By evaluating the component form, we can then recover from (A.32) exactly the form in (5.15a).

We now consider the spin-torsion equation. The spin tensor density of matter is defined

$$\sigma(a) \equiv -\delta_{\Omega(a)} \int |\mathrm{d}^4 x| \det \mathsf{h}^{-1} L_{\mathrm{M}}, \quad \sigma^{\mu}{}_{ij} \equiv (\gamma_i \wedge \gamma_j) \cdot \sigma(\mathsf{e}^{\mu}). \tag{A.33}$$

Using the results $\partial_{\Omega(q)} \langle \mathcal{T}(a) \mathcal{B} \rangle = \big(\bar{\mathsf{h}}(q) \cdot \mathcal{B} \big) \wedge a$ and $\partial_{\Omega(q),p} \langle \mathcal{R}(a \wedge b) \mathcal{B} \rangle = (q \wedge p) \cdot \underline{\mathsf{h}}(a \wedge b) \mathcal{B}$, and once again keeping careful track of the connection, we find that the spin-torsion equation is

$$\sigma(a) = \pi_{\mathcal{R}}(\bar{\mathsf{h}}(a \wedge \overleftrightarrow{D})) + \big(\bar{\mathsf{h}}(a) \cdot \pi_{\mathcal{T}}(\partial_b) \big) \wedge b. \tag{A.34}$$

Once again, the component form of (A.34) is found to be identical to (5.15b). For completeness we provide also the field equations (5.19) which stem from the multipliers themselves

$$\acute{\beta}_1 \mathcal{T}(a) + \acute{\beta}_2 a \cdot \partial_b \wedge \mathcal{T}(b) - \acute{\beta}_3 a \wedge \Big(\partial_b \cdot \mathcal{T}(b) \Big) = 0, \tag{A.35a}$$

$$\begin{aligned} & 2\acute{\alpha}_1 (a \wedge b) \mathcal{R} + \acute{\alpha}_2 \big[\mathcal{R}(a) \wedge b + a \wedge \mathcal{R}(b) \big] + \acute{\alpha}_3 \big[\bar{\mathcal{R}}(a) \wedge b + a \wedge \bar{\mathcal{R}}(b) \big] \\ & + 2\acute{\alpha}_4 \mathcal{R}(a \wedge b) + \acute{\alpha}_5 \big[\partial_c \wedge \mathcal{R}(c \wedge a) \cdot b - a \cdot \partial_c \wedge \mathcal{R}(c \wedge b) \big] + 2\acute{\alpha}_6 \bar{\mathcal{R}}(a \wedge b) = 0. \end{aligned} \tag{A.35b}$$

Thus, the main results of this appendix are the field equations (A.32) and (A.34), cast in terms of the generalised momenta (A.29a) and (A.29b).

Appendix B

Quadratic gravity

B.1 Spin projection operators

In this appendix we provide the SPOs which are used to decompose the field $\mathcal{A}_{ijk}$ in previous treatments of the PGT [156, 152, 153]. This decomposition is the basis of the particle spectra provided in Tables 2.1 and 4.1, and should be used as a reference when comparing with the alternative ADM J^P decomposition which we develop across Chapters 4 and 5. The SPOs generically take the form $\mathcal{P}_{\mathrm{XY}}(J^P)$, where the six Roman indices are suppressed and X and Y label independent sectors with the same J^P. In particular, the diagonal elements $\mathrm{X} = \mathrm{Y}$ form a complete set over all J^P sectors in $\mathcal{A}_{ijk}$, and $\mathrm{X} \neq \mathrm{Y}$ is only possible within the 1^- and 1^+ sectors, since the direct sum contains *two* independent representations of these J^P. For much of this thesis, we are working at the level of the torsion rather than the spin connection. Within the linearised regime set out in (2.32), if the derivatives of the translational gauge fields are ignored then $\mathcal{T}^i_{\ jk}$ and $\mathcal{A}^{ij}_{\ \ k}$ are related by the contortion $\mathcal{T}_{pqr} = \mathcal{N}_{pqr}{}^{ijk}\mathcal{A}_{ijk}$, where $\mathcal{N}_{pqr}{}^{ijk} \equiv 2\delta^j_p\delta^i_{[r}\delta^k_{q]}$. Thus all freedoms in the spin connection are inherited by the torsion. It is natural that the J^P sectors of one field map onto the other, indeed generally we find

$$\mathcal{N}_{pqr}{}^{ijk}\mathcal{P}_{\mathrm{XX}}(J^P)_{ijk}{}^{uvw}\mathcal{A}_{uvw} = \mathcal{P}_{\mathrm{XX}}(J^P)_{pqr}{}^{ijk}\mathcal{T}_{kji}\ . \tag{B.1}$$

Some nuance is however required in the case of the pseudovector tordion triplet, since $\mathcal{N}$ does not commute with $\mathcal{P}_{\mathrm{XY}}(1^+)$. The correct mixing in this case is given by the off-diagonal SPOs

$$\begin{aligned}
\mathcal{N}_{pqr}{}^{ijk}\mathcal{P}_{11}(1^+)_{ijk}{}^{uvw}\mathcal{A}_{uvw} &= \left(\mathcal{P}_{22}(1^+)_{pqr}{}^{ijk} - \frac{1}{\sqrt{2}}\mathcal{P}_{12}(1^+)_{pqr}{}^{ijk}\right)\mathcal{T}_{kji}\ ,\\
\mathcal{N}_{pqr}{}^{ijk}\mathcal{P}_{22}(1^+)_{ijk}{}^{uvw}\mathcal{A}_{uvw} &= \left(\mathcal{P}_{11}(1^+)_{pqr}{}^{ijk} + \frac{1}{\sqrt{2}}\mathcal{P}_{12}(1^+)_{pqr}{}^{ijk}\right)\mathcal{T}_{kji}\ .
\end{aligned} \tag{B.2}$$

With (B.1) and (B.2) in mind, it is therefore possible to consider J^P tordions as well-defined excitations of the torsion and/or the spin connection, though the latter is more conventional from the perspective of quantisation.

The building blocks of the SPOs are two k^i-dependent projections $\Omega^{ij} \equiv k^i k^j/k^2$ and $\Theta^{ij} \equiv \eta^{ij} - k^i k^j/k^2$. For the $\mathcal{A}_{ijk}$-field, the diagonal SPOs then have the following fundamental definitions

$$
\begin{aligned}
\grave{\mathcal{P}}_{11}(0^-)_{ijkpqr} &\equiv \frac{2}{3}\Theta_{ir}\Theta_{jp}\Theta_{kq} + \frac{1}{3}\Theta_{jp}\Theta_{jq}\Theta_{kr}, & \grave{\mathcal{P}}_{11}(0^+)_{ijkpqr} &\equiv \frac{2}{3}\Theta_{rq}\Theta_{kj}\Omega_{ip},\\
\grave{\mathcal{P}}_{11}(1^+)_{ijkpqr} &\equiv \Theta_{ir}\Theta_{kq}\Omega_{jp} + \Theta_{ip}\Theta_{kr}\Omega_{jq}, & \grave{\mathcal{P}}_{11}(1^-)_{ijkpqr} &\equiv \frac{2}{3}\Theta_{rq}\Theta_{ip}\Theta_{kj},\\
\grave{\mathcal{P}}_{11}(2^-)_{ijkpqr} &\equiv \frac{2}{3}\Theta_{ir}\Theta_{jq}\Omega_{kp} + \frac{2}{3}\Theta_{ip}\Theta_{jq}\Omega_{kr} - \Theta_{rq}\Theta_{ip}\Omega_{kj}, & \grave{\mathcal{P}}_{22}(1^-)_{ijkpqr} &\equiv 2\Theta_{ip}\Theta_{rq}\Theta_{kj},\\
\grave{\mathcal{P}}_{11}(2^+)_{ijkpqr} &\equiv -\frac{2}{3}\Theta_{rb}\Theta_{kj}\Omega_{ip} + \Theta_{ir}\Theta_{kp}\Omega_{jq} + \Theta_{ip}\Theta_{kr}\Omega_{jq}, & \grave{\mathcal{P}}_{22}(1^+)_{ijkpqr} &\equiv \Theta_{ip}\Theta_{jq}\Omega_{jr}.
\end{aligned}
\tag{B.3}
$$

Since the $\mathcal{A}_{ijk}$ field has two 1^+ and 1^- sectors, there is the opportunity for internal mixing. In particular the off-diagonal SPOs which are relevant for this work are $\grave{\mathcal{P}}_{12}(1^+)_{ijkpqr} \equiv -\sqrt{2}\Theta_{jp}\Theta_{kq}\Omega_{ir}$ and $\grave{\mathcal{P}}_{21}(1^+)_{ijkpqr} \equiv -\sqrt{2}\Theta_{qi}\Theta_{kj}\Omega_{ir}$. The diagonal SPOs are complete, idempotent and orthogonal across J^P sectors. The correctly symmetrised forms of all SPOs are given by $\mathcal{P}_{\mathrm{XY}}(J^P)_{ijkpqr} \equiv \grave{\mathcal{P}}_{\mathrm{XY}}(J^P)_{[ij]k[pq]r}$. For the simpler SPOs of the translational gauge fields, see [152] and references therein. We can also take the opportunity to notice that the two cosmological freedoms in the torsion tensor follow from the 0^+ and 0^- modes, as mentioned throughout this thesis, $\mathcal{P}_{11}(0^+)^{pij}{}_{qrk}\mathcal{T}^k{}_{ij} = \frac{2}{3}(\hat{\boldsymbol{e}}_t)^v U \delta^p{}_{[r}\eta_{q]v}$ and $\mathcal{P}_{11}(0^-)^{pij}{}_{qrk}\mathcal{T}^k{}_{ij} = -(\hat{\boldsymbol{e}}_t)^v Q \epsilon^p{}_{vqr}$. These expressions have their ADM equivalents in the first paragraph of Section 4.5, or in (5.28).

B.2 Cosmological torsion

In this appendix we obtain the form of cosmological torsion, as given in (2.37). This form was first rigorously identified by Tsamparlis [129], and has been used by both Boehmer and Bronowski [196] and Brechet, Hobson and Lasenby [197] in the study of cosmologies filled with Weyssenhoff fluids. One may arrive at (2.37) by noting that, under the SCP, the spacetime contains six global Killing vector fields $\mathcal{K}^i$, each tangent to the local Cauchy surface. Furthermore, cosmic fluids share a global, normalised velocity field $u^i = n^i$ (i.e. equated with the ADM vector), to which the Cauchy surfaces are orthogonal $n^i\mathcal{K}_i = 0$. We can use this to define the intrinsic metric on the Cauchy surfaces, which is also a projection tensor with vanishing Lie derivative

$$
\eta_{\overline{ij}} \equiv \eta_{ij} - n_i n_j, \quad \mathcal{L}_{\mathcal{K}}\eta_{\overline{ij}} = 0,
\tag{B.4}
$$

along with the projection $\mathcal{F}^{\overline{\acute{v}}}$ of any tensor $\mathcal{F}^{\acute{v}}$ (where we combine the variable rank and ADM projection notation from Chapters 4 and 5), and its projected covariant derivative $\mathcal{D}^{\parallel}{}_{\bar{i}}\mathcal{F}^{\acute{u}} \equiv \delta^{\overline{\acute{u}}}_{\overline{\acute{v}}}\mathcal{D}_{\bar{i}}\mathcal{F}^{\acute{v}}$. Our fundamental requirement is that $\mathcal{L}_{\mathcal{K}}\mathcal{T}^i{}_{jk} = 0$, but by (B.4) we must have $\mathcal{L}_{\mathcal{K}}\mathcal{T}^{\bar{i}}{}_{\overline{jk}} = 0$ also. Examining this, we find

$$
\mathcal{K}^{\bar{l}}\mathcal{D}^{\parallel}{}_{\bar{l}}\mathcal{T}^{\bar{i}}{}_{\overline{jk}} = \mathcal{T}^{\bar{l}}{}_{\overline{jk}}\,\mathcal{D}^{\parallel}{}_{\bar{l}}\mathcal{K}^{\bar{i}} - \left(\mathcal{T}^{\bar{i}}{}_{\overline{lk}}\,\mathcal{D}^{\parallel}{}_{\bar{j}} + \mathcal{T}^{\bar{i}}{}_{\overline{jl}}\,\mathcal{D}^{\parallel}{}_{\bar{k}}\right)\mathcal{K}^{\bar{l}} = \left(\eta^{\overline{pi}}\mathcal{T}^{\bar{l}}{}_{\overline{jk}} + \delta^{\bar{p}}_{\bar{j}}\mathcal{T}^{\overline{il}}{}_{\bar{k}} + \delta^{\bar{p}}_{\bar{k}}\mathcal{T}^{\bar{i}}{}_{\bar{j}}{}^{\bar{l}}\right)\mathcal{D}^{\parallel}{}_{[\bar{l}}\mathcal{K}_{\bar{p}]}.
\tag{B.5}
$$

There is freedom in the choice of the $\mathcal{K}^i$ to set to zero either side of (B.5). Doing so on the RHS enforces spatial homogeneity, so that the components $\mathcal{T}^i{}_{jk}$ are functions only of the coordinate t. On the LHS, we enforce isotropy, so that $\delta^{[\bar{p}}_{\bar{i}}s^{\bar{q}]}_{\bar{r}}\mathcal{T}^{\bar{r}}{}_{\overline{jk}} + \delta^{[\bar{p}}_{\bar{j}}\delta^{\bar{q}]}_{\bar{r}}\mathcal{T}_{\bar{i}}{}^{\bar{r}}{}_{\bar{k}} + \delta^{[\bar{p}}_{\bar{k}}\delta^{\bar{q}]}_{\bar{r}}\mathcal{T}_{\overline{ij}}{}^{\bar{r}} = 0$. From here we arrive at the pair of projected component constraints $\mathcal{T}^{\bar{i}}{}_{\overline{ji}} = 0$ and $\mathcal{T}_{\overline{ijk}} = \mathcal{T}_{\overline{[ijk]}}$, and by inspection we see that these admit only the form set out in (2.37).

B.3 Alternative cosmologies

We initially proposed that Class ^{3}C defined by Eqs. (2.52), (2.53) and (2.56), be refined to Class ^{3}C* by the final constraint (2.65) in order to satisfy correspondence with flat GR. In this appendix we investigate alternative constraints appearing in Fig. 2.1 which alter the particle content of the theory.

Class ^{4}H: k-screened dynamically open. An additional constraint

$$\sigma_2 = 0, \tag{B.6}$$

focuses Class ^{3}C onto Class ^{4}H. This is the cosmic class of Case 14 which may admit massless 2^+ gravitons as with Case 16, and also of Case 8, though the massless graviton in this case is not expected to be 2^+. Furthermore, (B.6) appears to have as profound a 'taming' effect on Class ^{3}C as the constraint (2.65) does. Since our analysis in (2.62) cannot be recycled to show this without a certain amount of difficulty, we will begin again from first principles. The cosmic implications of the quadratic Riemann–Cartan sector in Class ^{4}H are characterised by the single parameter σ_1, and those of the quadratic torsion by υ_2. The latter generally maintains the broken cosmological NSI, allowing for matter as a cosmic fluid. The cosmological E.o.M are significantly simplified by defining two fields from the observable torsion quantities, Φ and Ψ of dimension eV

$$\Psi \equiv \frac{\upsilon_2 U}{4\sqrt{3}\sigma_1 \kappa Q^2} - \frac{\sqrt{3}\partial_t Q}{Q}, \quad \Phi \equiv \Psi - \frac{U}{\sqrt{3}}. \tag{B.7}$$

The density balance equation now adopts the form of the first Friedmann equation (2.58), where the dimensionless densities of the torsion fields are Λ-like, in that $\rho_\Phi = -\kappa^{-1}\Phi^2$ and $\rho_\Psi = \kappa^{-1}\Psi^2$ are incorporated as $\Omega_\Phi \equiv \upsilon_2\kappa\rho_\Phi/3H^2$ and $\Omega_\Psi \equiv \upsilon_2\kappa\rho_\Psi/3H^2$. This relabeling becomes meaningful when we apply it to the torsion equations (2.51a) and (2.51b) which, if $Q \neq 0$, become respectively

$$\Psi = \sqrt{3}H, \quad \partial_t\Phi + H\Phi = 0. \tag{B.8}$$

These allow us to express (2.58) in terms of R, H and various constants, thereby encoding the curvature evolution. Specifically, we have from (B.8) $\Phi = \chi/R$, where χ is a constant of integration, so that the density equation reduces to

$$\Omega_\mathrm{r} + \Omega_\mathrm{m} + \Omega_\Lambda - \frac{\upsilon_2\chi^2}{3H^2R^2} = -\upsilon_2. \tag{B.9}$$

Given the *same* inequality constraint (2.67) that was applied to the root theory when constrained by (2.65), and noting that for Φ be an observable quantity we must have $\chi^2 \geq 0$, we have again uncovered emergent GR evolution, but now with a *strictly negative effective* k. This, in a theory that is fundamentally k-screened, results in dynamically *open* but geometrically *arbitrary* cosmology. It remains only to examine the evolution of the observable torsion quantities U and Q. We find $U = 3H - \sqrt{3}\chi/R$ and $\kappa Q^2 = \upsilon_2/4\sigma_1 - (\upsilon_2\chi/2\sqrt{3}\sigma_1 R^2)\int \mathrm{d}tR$. This is also similar to the torsion evolution in classes Class ^{3}C and Class ^{3}C*: U diverges on the approach to the radiation dominated Big Bang, while Q converges.

Class ^{4}I: power-law inflation. An alternative constraint to (2.65) is (2.55): this acts on the torsion rather than the Riemann–Cartan curvature sector – eliminating the former entirely. This constraint defines Class ^{4}I, of Case ^{*4}11 which again contains a propagating massless, potentially 2^+ graviton and also has gauge-invariant PCR. An undesirable side effect of (2.55) is the introduction of cosmological NSI. Nonetheless, we repeat the procedure used for Class ^{4}H by redefining (B.7) as

$$\Psi \equiv \frac{1}{\sigma_2 - \sigma_1}\left(\frac{\sigma_2 U}{\sqrt{3}} + \frac{\sigma_1\sqrt{3}\partial_t Q}{Q}\right), \quad \Phi \equiv \frac{\sigma_2\sqrt{\kappa}}{\sigma_2 - \sigma_1}\left(\frac{QU}{\sqrt{3}} + \sqrt{3}\partial_t Q\right). \tag{B.10}$$

This time, the Ψ field does not appear in the density balance equation, and the only possible source fluid is naturally NSI radiation $\Omega_{\mathrm{r}} + \Omega_\Phi = 0$. The coupling constant is also redefined according to $\Omega_\Phi \equiv (4\sigma_1^2 - \sigma_2^2)\kappa\rho_\Phi/3\sigma_2 H^2$. From Class ^{4}H we find that the second equality in (B.8) is slightly modified to $\partial_t\Phi + 2H\Phi = 0$, which translates to another effective radiation component. The curvature evolution is thus determined by the remaining torsion equations, which may be solved to give $U = 0$, $\partial_t Q/Q = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)t$ and $H = \sigma_2/(\sigma_1 + \sigma_2)t$, implying a potentially inflationary expansion, according to a power-law (see e.g. [261]) which depends on the theory parameters.

Class ^{5}M The final combination of (B.6) with (2.55) results in Class ^{5}M. While Case ^{*3}10 (unlike Case ^{*1}9) again may contain a massless 2^+ graviton and has the gauge-invariant PCR property, the cosmology is even more impoverished than Class ^{4}I, and we will not discuss it further. We will stop short of generalising the Φ-Ψ formalism in reverse to Class ^{3}C or repeating the analysis of (2.62) with conformally transformed ς so as to better accommodate Class ^{4}H. This concludes the summary of the child theories of Class ^{3}C.

Class ^{3}E: cyclic cosmologies In focusing on Class ^{3}C in Section 2.5.5 and its child cosmologies in Section 2.5.5 and in this appendix, we have neglected the parent Class ^{2}A and siblings Class ^{3}E and Class ^{3}D. The particle content of Case 15 of Class ^{3}D is similar to that of Case 16 of Class ^{3}C, with a potential massless 2^+ graviton. Indeed, Class ^{3}D and its child Class ^{4}J are good candidates for further investigation. In this section, we will very briefly focus on Class ^{3}E, which instead has a similar particle content to the parent cosmology, Class ^{2}A. Both classes are richly populated by critical cases with massive 0^- gravitons, though Case 1 in Class ^{3}E may additionally contain a massless 2^+ graviton.

In particular, we will retain the fundamentals of a k-screened Yang–Mills theory with (2.52) and (2.53), but instead of (2.56) we will enforce (2.55). To highlight the emergent inflationary effects we will set $\Lambda = 0$, admitting radiation and matter only. As a k-screened theory, the formula (2.57) still allows us to solve for U in terms of Q and H. The usual energy balance equations are no longer especially insightful, and so we work again at the level of the dynamical variables. We first see that (2.51c) allows Q to be expressed purely in terms of the matter content $Q^2 = H^2\Omega_{\mathrm{m}}/2\upsilon_1$. By substituting this and (2.57) into (2.51d) we then obtain the following solution

$$a = c_1\left(\cosh(c_2 t) - 1\right), \tag{B.11}$$

recalling $a \equiv R/R_0$, where the amplitude depends on the ratio of radiation to matter, and the characteristic time on the cosmic theory parameters

$$c_1 \equiv \Omega_{\mathrm{r},0}/\Omega_{\mathrm{m},0}, \quad \kappa c_2^2 \equiv \sigma_2\upsilon_1/(\sigma_2^2 - 4\sigma_1^2). \tag{B.12}$$

Thus, through a suitable choice of the theory parameters we may obtain either cyclic Universes in which the Big Crunch and Big Bang are periodic, or perpetual exponential inflation to the Future Conformal Boundary.

Our analysis in Chapter 2 of each cosmic class relies only on the defining equalities of critical cases, but we must also consider the accompanying unitarity inequalities in Table 2.1. Of greatest concern is Class ^{3}C. We find the relevant inequality constraint on Case 16 reduces to

$$(3r_5 + 2\sigma_1 + \sigma_2)(3r_5 + 8\sigma_1 + 4\sigma_2)(2\sigma_1 + \sigma_2) < 0, \tag{B.13}$$

from which r_5 cannot be eliminated in favor of σ_1, σ_2 or υ_2. This means that the unitarity of Case 16 does not constrain the cosmological picture of Class ^{3}C or Class ^{3}C* discussed in Section 2.5.5: we will return to the unitarity of these theories in Chapter 5.

Of the other child cosmologies of Class ^{3}C examined above, we find that unitarity of Case ^{*4}11 of the cosmologically NSI Class ^{4}I also requires (B.13), while the more promising Class ^{4}H requires

$$\sigma_1(3r_5 + 2\sigma_1)(3r_5 + 8\sigma_1) < 0, \tag{B.14}$$

for the unitarity of Case 14 – once more r_5 cannot be expressed in terms of σ_1 or υ_2. The other cosmologically NSI Class ^{5}M also requires (B.14) for the unitarity of Case ^{*3}10. Although not considered here, we note that the promising Case 15 and Case 12 respectively of Class ^{3}D and Class ^{4}J also require (B.14).

In fact, the unitarity inequalities only begin to impinge on the cosmology when the massive 0^- mode propagates. We touched already on Class ^{3}E, of which Case 1 also requires (B.13), and two additional inequalities

$$\sigma_2 < 0, \quad \upsilon_1 < 0. \tag{B.15}$$

Although these explicitly affect the cosmic theory parameters remaining to Class ^{3}E, they do not fully constrain the characteristic time (B.12) of the hyperbolic solution we consider in (B.11). We will not examine Case 27, Case ^{*7}30 or Case ^{*9}35 of Class ^{3}E, since they do not contain massless particles. The constraints (B.15) become very important when we generalise Class ^{3}C to Class ^{2}A in Chapter 3, where they ensure that the theory introduces a *positive* effective cosmological constant.

B.4 Comparison with the literature

Given the popularity of ten-parameter PGT^{q+} cosmology mentioned in Section 2.1, it is necessary to attempt in this appendix some comparison with the literature, though we will not consider extension to the odd-parity sector discussed by [145, 144, 147, 127, 146].

The original paper by Minkevich [130] only admits U, and not Q on the grounds of spacetime parity – an examination of Eqs. (2.51a) to (2.51d) indicates that σ_1 and σ_2 do not arise in this case, and so k-screening cannot meaningfully occur. Furthermore, [130] retains $\check{\alpha}_0$ in order to force the correspondence principle. We note that this situation is slightly complicated in [131, 133, 137] by the extension to MAGT. In [134–136] it appears that both U and Q are incorporated, but we find that the two constraints imposed on (2.26) translate to (2.55), while $\check{\alpha}_0$ remains free. Throughout [148, 149] we again believe[1] $\check{\alpha}_0$

[1] In comparison to these papers we use the identity $(\epsilon_{ijkl}\mathcal{R}^{ijkl})^2 = 4\mathcal{R}_{ijkl}(4\mathcal{R}^{ikjl} - \mathcal{R}^{ijkl} - \mathcal{R}^{klij})$.

to be retained, with (2.55) imposed at certain points. Within [148] two further constraints are applied which reduce to

$$\sigma_1 - \sigma_3 = \sigma_2 - \sigma_3 = 0. \tag{B.16}$$

Thus, while σ_3 remains free, (B.16) implies the final constraint (2.65) which separates Class ^{3}C* from Class ^{3}C. Precisely (B.16) is applied in [141], along with the torsion constraint

$$4\upsilon_1 + \upsilon_2 = 0, \tag{B.17}$$

to define the original SNY lagrangian. We note that (B.17) itself features in Fig. 2.1 to distinguish Class ^{4}L from Class ^{3}F. The SNY generalisation studied in [143] replaces Eq. (B.16) with

$$\sigma_2 + 2\sigma_1 - 3\sigma_3 = 0, \tag{B.18}$$

though we do not believe the quadratic torsion sector to be constrained. Once again, (B.18) features in Fig. 2.1 to distinguish Class ^{3}G from Class ^{2}B.

Finally, we will consider [140], in which a solution to the cosmological equations of RST was presented. Here we will show that the solution satisfies a much broader class of cosmologically NSI theories. Beginning from the original root theory, we restrict to Yang–Mills gravity by applying (2.52), and then to cosmologically NSI gravity by eliminating the torsion with (2.56) and (2.55). The quadratic Riemann sector is then refined with two new constraints

$$\sigma_1 = \sigma_2 - 3\sigma_3 = 0. \tag{B.19}$$

The cosmic class to which RST belongs is not populated by any of the critical cases considered here, and as such it does not appear in Fig. 2.1. Note however, that it can be considered a child of Class ^{3}G, which appears only to contain critical cases with massive 0^- gravitons. The torsion equations (2.51a) and (2.51b) then take the form

$$\left(\delta\tilde{\mathcal{L}}_T/\delta X\right)_{\mathrm{F}} \propto \partial_\tau^2 X + 2X(3Y^2/4 - X^2 - k), \quad \left(\delta\tilde{\mathcal{L}}_T/\delta Y\right)_{\mathrm{F}} \propto -\partial_\tau^2 Y + 2Y(3X^2 - Y^2/4 + k), \tag{B.20}$$

in which their mutual symmetry – first noted in Section 2.5.3 – is made apparent. This can be exploited by encoding both equations as $\partial_\tau^2 Z - 2Z^3 + 2kZ = 0$ though a complex torsion variable $Z \equiv X + iY/2$. The single resulting equation can then be solved compactly for Z in terms of the Weierstrass elliptic function, such that the source ϱ_{r} appears as a constant of integration. This compact solution describes an interesting Universe, in which the Hubble number and torsion may evolve chaotically. If we set $U = Q = 0$, then the density equation analogous to (2.58) becomes $\Omega_{\mathrm{r}} + (8\sigma_2\kappa/3)\big((\partial_t H/H)^2 + 2\partial_t H - H^2\Omega_k(\Omega_k - 2)\big) = 0$, from which $\partial_t H$ can then be eliminated by the observable form of (2.51a) $\partial_t^2 H + 4H\partial_t H + 2H^3\Omega_k = 0$. By writing the implied integration constant as an effective cosmological constant, Λ, this becomes $\partial_t H = H^2(\Omega_k - 2) + \frac{2}{3}\Lambda$. The final density equation then looks quite familiar $9\Omega_{\mathrm{r}}/8\kappa\Lambda + \Omega_\Lambda + \Omega_k = 1$, as an effective cosmological constant emerges up to a renormalisation of the radiation density.

B.5 Cosmological equations of Class ^{3}C

In this appendix we provide the modified gravitational densities in (2.58) and the coefficients to the auxiliary torsion equation (2.60) as follows

$$\begin{aligned}\Omega_\Psi+\Omega_\Phi=&\frac{\left(\left(16\,{\sigma_1}^2-4\,{\sigma_2}^2\right)\kappa^2Q^2+\kappa\sigma_2\,\upsilon_2\right)\partial_tQ^2}{\left(4\,Q^2\sigma_2\,\kappa-\upsilon_2\right)H^2}\\&+32\,\frac{Q\left(\kappa\left({\sigma_1}^2-1/4\,{\sigma_2}^2\right)Q^2-1/4\,\upsilon_2\,\left(\sigma_1-\sigma_2/4\right)\right)\kappa\partial_tQ}{\left(4\,Q^2\sigma_2\,\kappa-\upsilon_2\right)H}\\&+16\,\frac{\left(\kappa\left({\sigma_1}^2-1/4\,{\sigma_2}^2\right)Q^2-1/2\,\left(\sigma_1-5/8\,\sigma_2\right)\upsilon_2\right)Q^2\kappa}{4\,Q^2\sigma_2\,\kappa-\upsilon_2},\end{aligned}\tag{B.21a}$$

$$f_1\equiv2\,Q\left(4\,\sigma_2\,\kappa Q^2-\upsilon_2\right)\left(16\,\kappa Q^2{\sigma_1}^2-4\,\kappa Q^2{\sigma_2}^2+\sigma_2\,\upsilon_2\right),\tag{B.21b}$$

$$f_2\equiv-32\,{\sigma_1}^2\upsilon_2\,\kappa Q^3,\tag{B.21c}$$

$$f_3\equiv6\,Q\left(4\,\sigma_2\,\kappa Q^2-\upsilon_2\right)\left(16\,\kappa Q^2{\sigma_1}^2-4\,\kappa Q^2{\sigma_2}^2+\sigma_2\,\upsilon_2\right),\tag{B.21d}$$

$$f_4\equiv2\,Q\left(4\,\sigma_2\,\kappa Q^2-\upsilon_2\right)\left(16\,\kappa Q^2{\sigma_1}^2-4\,\kappa Q^2{\sigma_2}^2-4\,\upsilon_2\,\sigma_1+\sigma_2\,\upsilon_2\right),\tag{B.21e}$$

$$\begin{aligned}f_5\equiv&\,256\,Q\Big(\left(\sigma_2\,\kappa^2{\sigma_1}^2-1/4\,{\sigma_2}^3\kappa^2\right)Q^4-1/8\,\left({\sigma_1}^2+3\,\sigma_1\,\sigma_2-{\sigma_2}^2\right)\upsilon_2\,\kappa Q^2\\&+1/32\,\left(\sigma_1+\sigma_2/2\right){\upsilon_2}^2\Big).\end{aligned}\tag{B.21f}$$

B.6 The quadratic couplings

We provide in this appendix comprehensive translations between the quadratic coupling formalisms used throughout Chapters 2 to 5. The Einstein–Hilbert coupling is given throughout as

$$\alpha_0\equiv\check{\alpha}_0\equiv\hat{\alpha}_0\equiv2\kappa l.\tag{B.22}$$

The quadratic couplings (with the order of the equation sets sometimes adjusted for a clear layout) are then

$$\begin{aligned}&\alpha_1\equiv\check{\alpha}_1, &&\alpha_2\equiv\check{\alpha}_2, &&\alpha_3\equiv\check{\alpha}_3,\\&\alpha_4\equiv\frac{\check{\alpha}_4}{2}-\frac{\check{\alpha}_5}{2}, &&\alpha_5\equiv\check{\alpha}_5, &&\alpha_6\equiv\frac{\check{\alpha}_6}{2},\\&\beta_1\equiv-\frac{\check{\beta}_1}{2}-\frac{\check{\beta}_2}{2}, &&\beta_2\equiv\check{\beta}_2, &&\beta_3\equiv\check{\beta}_3,\end{aligned}\tag{B.23a}$$

$$\begin{aligned}&\beta_1\equiv\frac{\kappa t_1}{3}+\frac{\kappa t_2}{12}+\frac{\kappa l}{4}, &&\beta_2\equiv\frac{\kappa t_1}{3}-\frac{\kappa t_2}{6}+\frac{\kappa l}{2}, &&\beta_3\equiv-\frac{\kappa t_1}{3}+\frac{2\kappa t_3}{3}-\kappa l,\\&\alpha_1\equiv r_6, &&\alpha_2\equiv r_4+r_5, &&\alpha_3\equiv r_4-r_5,\\&\alpha_4\equiv\frac{r_1}{3}+\frac{r_2}{6}, &&\alpha_5\equiv\frac{2r_1}{3}-\frac{2r_2}{3}, &&\alpha_6\equiv\frac{r_1}{3}+\frac{r_2}{6}-r_3,\end{aligned}\tag{B.23b}$$

$$\begin{aligned}&\alpha_1\equiv\frac{\hat{\alpha}_1}{3}-\frac{\hat{\alpha}_4}{2}+\frac{\hat{\alpha}_6}{6}, &&\alpha_2\equiv-\hat{\alpha}_1-\hat{\alpha}_2+\hat{\alpha}_4+\hat{\alpha}_5, &&\alpha_3\equiv-\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_4-\hat{\alpha}_5,\\&\alpha_4\equiv\frac{\hat{\alpha}_1}{3}+\frac{\hat{\alpha}_2}{2}+\frac{\hat{\alpha}_3}{6}, &&\alpha_5\equiv\frac{2\hat{\alpha}_1}{3}-\frac{2\hat{\alpha}_3}{3}, &&\alpha_6\equiv\frac{\hat{\alpha}_1}{3}-\frac{\hat{\alpha}_2}{2}+\frac{\hat{\alpha}_3}{6},\\&\beta_1\equiv\frac{2\hat{\beta}_1}{3}+\frac{\hat{\beta}_3}{3}, &&\beta_2\equiv\frac{2\hat{\beta}_1}{3}-\frac{2\hat{\beta}_3}{3}, &&\beta_3\equiv-\frac{2\hat{\beta}_1}{3}+\frac{2\hat{\beta}_2}{3},\end{aligned}\tag{B.23c}$$

$$\breve{\alpha}_1 \equiv \alpha_1, \quad \breve{\alpha}_2 \equiv \alpha_2, \quad \breve{\alpha}_3 \equiv \alpha_3,$$
$$\breve{\alpha}_4 \equiv 2\alpha_4 + \alpha_5, \quad \breve{\alpha}_5 \equiv \alpha_5, \quad \breve{\alpha}_6 \equiv 2\alpha_6,$$
$$\breve{\beta}_1 \equiv -2\beta_1 - \beta_2, \quad \breve{\beta}_2 \equiv \beta_2, \quad \breve{\beta}_3 \equiv \beta_3, \tag{B.23d}$$

$$\breve{\beta}_1 \equiv -\kappa l - \kappa t_1, \quad \breve{\beta}_2 \equiv \frac{\kappa t_1}{3} - \frac{\kappa t_2}{6} + \frac{\kappa l}{2}, \quad \breve{\beta}_3 \equiv -\frac{\kappa t_1}{3} + \frac{2\kappa t_3}{3} - \kappa l,$$
$$\breve{\alpha}_1 \equiv r_6, \quad \breve{\alpha}_2 \equiv r_4 + r_5, \quad \breve{\alpha}_3 \equiv r_4 - r_5,$$
$$\breve{\alpha}_4 \equiv \frac{4r_1}{3} - \frac{r_2}{3}, \quad \breve{\alpha}_5 \equiv \frac{2r_1}{3} - \frac{2r_2}{3}, \quad \breve{\alpha}_6 \equiv \frac{2r_1}{3} + \frac{r_2}{3} - 2r_3, \tag{B.23e}$$

$$\breve{\alpha}_1 \equiv \frac{\hat{\alpha}_1}{3} - \frac{\hat{\alpha}_4}{2} + \frac{\hat{\alpha}_6}{6}, \quad \breve{\alpha}_2 \equiv -\hat{\alpha}_1 - \hat{\alpha}_2 + \hat{\alpha}_4 + \hat{\alpha}_5, \quad \breve{\alpha}_3 \equiv -\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_4 - \hat{\alpha}_5,$$
$$\breve{\alpha}_4 \equiv \frac{4\hat{\alpha}_1}{3} + \hat{\alpha}_2 - \frac{\hat{\alpha}_3}{3}, \quad \breve{\alpha}_5 \equiv \frac{2\hat{\alpha}_1}{3} - \frac{2\hat{\alpha}_3}{3}, \quad \breve{\alpha}_6 \equiv \frac{2\hat{\alpha}_1}{3} - \hat{\alpha}_2 + \frac{\hat{\alpha}_3}{3},$$
$$\breve{\beta}_1 \equiv -2\hat{\beta}_1, \quad \breve{\beta}_2 \equiv \frac{2\hat{\beta}_1}{3} - \frac{2\hat{\beta}_3}{3}, \quad \breve{\beta}_3 \equiv -\frac{2\hat{\beta}_1}{3} + \frac{2\hat{\beta}_2}{3}, \tag{B.23f}$$

$$\kappa t_1 \equiv 2\beta_1 + \beta_2 - \frac{\alpha_0}{2}, \quad \kappa t_2 \equiv 4\beta_1 - 4\beta_2 + \frac{\alpha_0}{2}, \quad \kappa t_3 \equiv \beta_1 + \frac{\beta_2}{2} + \frac{3\beta_3}{2} + \frac{\alpha_0}{2},$$
$$r_1 \equiv 2\alpha_4 + \frac{\alpha_5}{2}, \quad r_2 \equiv 2\alpha_4 - \alpha_5, \quad r_3 \equiv \alpha_4 - \alpha_6,$$
$$r_4 \equiv \frac{\alpha_2}{2} + \frac{\alpha_3}{2}, \quad r_5 \equiv \frac{\alpha_2}{2} - \frac{\alpha_3}{2}, \quad r_6 \equiv \alpha_1, \tag{B.23g}$$

$$\kappa t_1 \equiv -\breve{\beta}_1 - \frac{\breve{\alpha}_0}{2}, \quad \kappa t_2 \equiv -2\breve{\beta}_1 - 6\breve{\beta}_2 + \frac{\breve{\alpha}_0}{2}, \quad \kappa t_3 \equiv -\frac{\breve{\beta}_1}{2} + \frac{3\breve{\beta}_3}{2} + \frac{\breve{\alpha}_0}{2},$$
$$r_1 \equiv \breve{\alpha}_4 - \frac{\breve{\alpha}_5}{2}, \quad r_2 \equiv \breve{\alpha}_4 - 2\breve{\alpha}_5, \quad r_3 \equiv \frac{\breve{\alpha}_4}{2} - \frac{\breve{\alpha}_5}{2} - \frac{\breve{\alpha}_6}{2},$$
$$r_4 \equiv \frac{\breve{\alpha}_2}{2} + \frac{\breve{\alpha}_3}{2}, \quad r_5 \equiv \frac{\breve{\alpha}_2}{2} - \frac{\breve{\alpha}_3}{2}, \quad r_6 \equiv \breve{\alpha}_1, \tag{B.23h}$$

$$r_1 \equiv \hat{\alpha}_1 + \hat{\alpha}_2, \quad r_2 \equiv \hat{\alpha}_2 + \hat{\alpha}_3, \quad r_3 \equiv \hat{\alpha}_2,$$
$$r_4 \equiv -\hat{\alpha}_1 + \hat{\alpha}_4, \quad r_5 \equiv -\hat{\alpha}_2 + \hat{\alpha}_5, \quad r_6 \equiv \frac{\hat{\alpha}_1}{3} - \frac{\hat{\alpha}_4}{2} + \frac{\hat{\alpha}_6}{6},$$
$$\kappa t_1 \equiv 2\hat{\beta}_1 - \frac{\hat{\alpha}_0}{2}, \quad \kappa t_2 \equiv 4\hat{\beta}_3 + \frac{\hat{\alpha}_0}{2}, \quad \kappa t_3 \equiv \hat{\beta}_2 + \frac{\hat{\alpha}_0}{2}, \tag{B.23i}$$

$$\hat{\alpha}_2 \equiv \alpha_4 - \alpha_6, \quad \hat{\alpha}_3 \equiv \alpha_4 - \alpha_5 + \alpha_6, \quad \hat{\alpha}_6 \equiv 6\alpha_1 + \frac{3\alpha_2}{2} + \frac{3\alpha_3}{2} + \alpha_4 + \frac{\alpha_5}{2} + \alpha_6,$$
$$\hat{\alpha}_1 \equiv \alpha_4 + \frac{\alpha_5}{2} + \alpha_6, \quad \hat{\alpha}_4 \equiv \frac{\alpha_2}{2} + \frac{\alpha_3}{2} + \alpha_4 + \frac{\alpha_5}{2} + \alpha_6, \quad \hat{\alpha}_5 \equiv \frac{\alpha_2}{2} - \frac{\alpha_3}{2} + \alpha_4 - \alpha_6,$$
$$\hat{\beta}_1 \equiv \beta_1 + \frac{\beta_2}{2}, \quad \hat{\beta}_2 \equiv \beta_1 + \frac{\beta_2}{2} + \frac{3\beta_3}{2}, \quad \hat{\beta}_3 \equiv \beta_1 - \beta_2, \tag{B.23j}$$

$$\hat{\alpha}_2 \equiv \frac{\breve{\alpha}_4}{2} - \frac{\breve{\alpha}_5}{2} - \frac{\breve{\alpha}_6}{2}, \quad \hat{\alpha}_3 \equiv \frac{\breve{\alpha}_4}{2} - \frac{3\breve{\alpha}_5}{2} + \frac{\breve{\alpha}_6}{2}, \quad \hat{\alpha}_6 \equiv 6\breve{\alpha}_1 + \frac{3\breve{\alpha}_2}{2} + \frac{3\breve{\alpha}_3}{2} + \frac{\breve{\alpha}_4}{2} + \frac{\breve{\alpha}_6}{2},$$
$$\hat{\alpha}_1 \equiv \frac{\breve{\alpha}_4}{2} + \frac{\breve{\alpha}_6}{2}, \quad \hat{\alpha}_4 \equiv \frac{\breve{\alpha}_2}{2} + \frac{\breve{\alpha}_3}{2} + \frac{\breve{\alpha}_4}{2} + \frac{\breve{\alpha}_6}{2}, \quad \hat{\alpha}_5 \equiv \frac{\breve{\alpha}_2}{2} - \frac{\breve{\alpha}_3}{2} + \frac{\breve{\alpha}_4}{2} - \frac{\breve{\alpha}_5}{2} - \frac{\breve{\alpha}_6}{2},$$

$$\hat{\beta}_1 \equiv -\frac{\check{\beta}_1}{2}, \qquad \hat{\beta}_2 \equiv -\frac{\check{\beta}_1}{2} + \frac{3\check{\beta}_3}{2}, \qquad \hat{\beta}_3 \equiv -\frac{\check{\beta}_1}{2} - \frac{3\check{\beta}_2}{2}, \tag{B.23k}$$

$$\begin{aligned} &\hat{\alpha}_1 \equiv r_1 - r_3, && \hat{\alpha}_2 \equiv r_3, && \hat{\alpha}_3 \equiv r_2 - r_3, \\ &\hat{\alpha}_4 \equiv r_1 - r_3 + r_4, && \hat{\alpha}_5 \equiv r_3 + r_5, && \hat{\alpha}_6 \equiv r_1 - r_3 + 3r_4 + 6r_6, \\ &\hat{\beta}_1 \equiv \frac{\kappa t_1}{2} + \frac{\kappa l}{2}, && \hat{\beta}_2 \equiv -\kappa l + \kappa t_3, && \hat{\beta}_3 \equiv \frac{\kappa t_2}{4} - \frac{\kappa l}{4}, \end{aligned} \tag{B.23l}$$

thus providing all possible combinations.

B.7 The cosmic couplings

We provide in this appendix translations of the cosmological couplings from Chapter 2 into the formalisms set out in Appendix B.6

$$\begin{aligned} &\sigma_1 \equiv \frac{3\alpha_1}{2} + \frac{\alpha_2}{4} + \frac{\alpha_3}{4} + \frac{\alpha_5}{4} - \frac{\alpha_6}{2}, && \sigma_2 \equiv \frac{3\alpha_1}{2} + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} + \frac{3\alpha_4}{2} - \frac{\alpha_5}{4} + \frac{\alpha_6}{2}, \\ &\sigma_3 \equiv \frac{3\alpha_1}{2} + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} + \frac{\alpha_4}{2} + \frac{\alpha_5}{4} + \frac{\alpha_6}{2}, && \upsilon_1 \equiv -2\beta_1 + 2\beta_2, \quad \upsilon_2 \equiv 2\beta_1 + \beta_2 + 3\beta_3, \end{aligned} \tag{B.24a}$$

$$\begin{aligned} &\sigma_1 \equiv \frac{3\check{\alpha}_1}{2} + \frac{\check{\alpha}_2}{4} + \frac{\check{\alpha}_3}{4} + \frac{\check{\alpha}_5}{4} - \frac{\check{\alpha}_6}{4}, && \sigma_2 \equiv \frac{3\check{\alpha}_1}{2} + \frac{\check{\alpha}_2}{2} + \frac{\check{\alpha}_3}{2} + \frac{3\check{\alpha}_4}{4} - \check{\alpha}_5 + \frac{\check{\alpha}_6}{4}, \\ &\sigma_3 \equiv \frac{3\check{\alpha}_1}{2} + \frac{\check{\alpha}_2}{2} + \frac{\check{\alpha}_3}{2} + \frac{\check{\alpha}_4}{4} + \frac{\check{\alpha}_6}{4}, && \upsilon_1 \equiv \check{\beta}_1 + 3\check{\beta}_2, \quad \upsilon_2 \equiv -\check{\beta}_1 + 3\check{\beta}_3, \end{aligned} \tag{B.24b}$$

$$\begin{aligned} &\sigma_1 \equiv -\frac{r_2}{4} + \frac{r_3}{2} + \frac{r_4}{2} + \frac{3r_6}{2}, && \sigma_2 \equiv \frac{r_1}{2} + \frac{r_2}{2} - \frac{r_3}{2} + r_4 + \frac{3r_6}{2}, \\ &\sigma_3 \equiv \frac{r_1}{2} - \frac{r_3}{2} + r_4 + \frac{3r_6}{2}, && \upsilon_1 \equiv -\frac{\kappa t_2}{2} + \frac{\kappa l}{2}, \quad \upsilon_2 \equiv -2\kappa l + 2\kappa t_3, \end{aligned} \tag{B.24c}$$

$$\begin{aligned} &\sigma_1 \equiv \frac{\hat{\alpha}_2}{4} - \frac{\hat{\alpha}_3}{4} - \frac{\hat{\alpha}_4}{4} + \frac{\hat{\alpha}_6}{4}, && \sigma_2 \equiv \frac{\hat{\alpha}_2}{2} + \frac{\hat{\alpha}_3}{2} + \frac{\hat{\alpha}_4}{4} + \frac{\hat{\alpha}_6}{4}, \\ &\sigma_3 \equiv \frac{\hat{\alpha}_4}{4} + \frac{\hat{\alpha}_6}{4}, && \upsilon_1 \equiv -2\hat{\beta}_3, \quad \upsilon_2 \equiv 2\hat{\beta}_2, \end{aligned} \tag{B.24d}$$

again providing all combinations.

B.8 The transfer couplings

We provide in this appendix translations of the transfer couplings from Chapter 5 into the formalisms set out in Appendix B.6. The first half of the transfer couplings in (5.11) are found to be

$$\begin{aligned} &\bar{\alpha}^{\|\|}_{0^+} \equiv \frac{1}{2}(\bar{\alpha}_4 + \bar{\alpha}_6), && \bar{\alpha}^{\|\|}_{0^-} \equiv \frac{1}{2}(\bar{\alpha}_2 + \bar{\alpha}_3), && \bar{\alpha}^{\|\|}_{1^+} \equiv -\frac{1}{2}(\bar{\alpha}_2 + \bar{\alpha}_5), && \bar{\alpha}^{\|\|}_{1^-} \equiv \frac{1}{2}(\bar{\alpha}_4 + \bar{\alpha}_5), \\ &\bar{\alpha}^{\|\|}_{2^+} \equiv \frac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_4), && \bar{\alpha}^{\|\|}_{2^-} \equiv \frac{1}{2}(\bar{\alpha}_1 + \bar{\alpha}_2), && \bar{\alpha}^{\perp\|}_{0^+} \equiv -\frac{1}{4}(\bar{\alpha}_4 - \bar{\alpha}_6), && \bar{\alpha}^{\perp\|}_{0^-} \equiv \frac{1}{2}(\bar{\alpha}_2 - \bar{\alpha}_3), \\ &\bar{\alpha}^{\perp\|}_{1^+} \equiv -\frac{1}{2}(\bar{\alpha}_2 - \bar{\alpha}_5), && \bar{\alpha}^{\perp\|}_{1^-} \equiv \frac{1}{2}(\bar{\alpha}_4 - \bar{\alpha}_5), && \bar{\alpha}^{\perp\|}_{2^+} \equiv \frac{1}{2}(\bar{\alpha}_1 - \bar{\alpha}_4), && \bar{\alpha}^{\perp\|}_{2^-} \equiv -\frac{1}{2}(\bar{\alpha}_1 - \bar{\alpha}_2), \end{aligned} \tag{B.25}$$

and the remaining couplings are *mostly* found using the rules $\bar{\alpha}^{\perp\perp}_{A}\equiv\frac{1}{2}\bar{\alpha}^{\|\|}_{A}$ and $\bar{\alpha}^{\|\perp}_{A}\equiv\frac{1}{2}\bar{\alpha}^{\perp\|}_{A}$, with the three exceptions $\bar{\alpha}^{\perp\perp}_{1^+}\equiv-\frac{1}{2}\bar{\alpha}^{\|\|}_{1^+}$, $\bar{\alpha}^{\|\perp}_{1^+}\equiv-\frac{1}{2}\bar{\alpha}^{\perp\|}_{1^+}$ and $\bar{\alpha}^{\perp\|}_{0^+}\equiv\frac{1}{2}\bar{\alpha}^{\|\perp}_{0^+}$, and these quirks just result from the 'human' normalisation of the SO(3) representations. It goes without saying that a precisely equivalent formulation can be constructed for the couplings $\{\hat{\alpha}_I\}$. The translational transfer couplings are

$$\begin{aligned}
&\bar{\beta}^{\|\|}_{0^+}\equiv 0, && \bar{\beta}^{\|\|}_{0^-}\equiv\frac{1}{6}\bar{\beta}_3, && \bar{\beta}^{\|\|}_{1^+}\equiv\frac{1}{3}(2\bar{\beta}_1+\bar{\beta}_3), && \bar{\beta}^{\|\|}_{1^-}\equiv\frac{1}{3}(\bar{\beta}_1+2\bar{\beta}_2),\\
&\bar{\beta}^{\|\|}_{2^+}\equiv 0, && \bar{\beta}^{\|\|}_{2^-}\equiv\bar{\beta}_1, && \bar{\beta}^{\perp\|}_{0^+}\equiv 0, && \bar{\beta}^{\perp\|}_{0^-}\equiv 0,\\
&\bar{\beta}^{\perp\|}_{1^+}\equiv-\frac{1}{3}(\bar{\beta}_1-\bar{\beta}_3), && \bar{\beta}^{\perp\|}_{1^-}\equiv-\frac{1}{3}(\bar{\beta}_1-\bar{\beta}_2), && \bar{\beta}^{\perp\|}_{2^+}\equiv 0, && \bar{\beta}^{\perp\|}_{2^-}\equiv 0,\\
&\bar{\beta}^{\perp\perp}_{0^+}\equiv\frac{1}{2}\bar{\beta}_2, && \bar{\beta}^{\perp\perp}_{0^-}\equiv 0, && \bar{\beta}^{\perp\perp}_{1^+}\equiv\frac{1}{6}(\bar{\beta}_1+2\bar{\beta}_3), && \bar{\beta}^{\perp\perp}_{1^-}\equiv\frac{1}{6}(2\bar{\beta}_1+\bar{\beta}_2),\\
&\bar{\beta}^{\perp\perp}_{2^+}\equiv\frac{1}{2}\bar{\beta}_1, && \bar{\beta}^{\perp\perp}_{2^-}\equiv 0,
\end{aligned}\tag{B.26}$$

where we find $\bar{\beta}^{\perp\|}_{E}\equiv\bar{\beta}^{\|\perp}_{E}$. We can thus summarise some very important relations for nonvanishing transfer couplings as

$$\frac{\bar{\alpha}^{\|\|}_{A}}{\bar{\alpha}^{\perp\perp}_{A}}\equiv\frac{\bar{\alpha}^{\perp\|}_{A}}{\bar{\alpha}^{\|\perp}_{A}}\equiv\frac{\hat{\alpha}^{\|\|}_{A}}{\hat{\alpha}^{\perp\perp}_{A}}\equiv\frac{\hat{\alpha}^{\perp\|}_{A}}{\hat{\alpha}^{\|\perp}_{A}}=2,\quad\frac{\bar{\beta}^{\perp\|}_{E}}{\bar{\beta}^{\|\perp}_{E}}\equiv\frac{\hat{\beta}^{\perp\|}_{E}}{\hat{\beta}^{\|\perp}_{E}}=1,\tag{B.27}$$

with two sets of exceptions in the rotational sector

$$\frac{\bar{\alpha}^{\|\|}_{1^+}}{\bar{\alpha}^{\perp\perp}_{1^+}}\equiv\frac{\bar{\alpha}^{\perp\|}_{1^+}}{\bar{\alpha}^{\|\perp}_{1^+}}\equiv\frac{\hat{\alpha}^{\|\|}_{1^+}}{\hat{\alpha}^{\perp\perp}_{1^+}}\equiv\frac{\hat{\alpha}^{\perp\|}_{1^+}}{\hat{\alpha}^{\|\perp}_{1^+}}=-2,\quad\frac{\bar{\alpha}^{\|\|}_{0^+}}{\bar{\alpha}^{\perp\perp}_{0^+}}\equiv\frac{\bar{\alpha}^{\|\perp}_{0^+}}{\bar{\alpha}^{\perp\|}_{0^+}}\equiv\frac{\hat{\alpha}^{\|\|}_{0^+}}{\hat{\alpha}^{\perp\perp}_{0^+}}\equiv\frac{\hat{\alpha}^{\|\perp}_{0^+}}{\hat{\alpha}^{\perp\|}_{0^+}}=2.\tag{B.28}$$

The resulting effect of the multipliers in the Lagrangian picture (5.19) translates to

$$\bar{\alpha}_1\neq 0\Rightarrow\underline{\mathcal{R}}_{\langle\bar{i}\bar{j}\rangle}+\mathcal{R}_{\perp\langle\bar{i}\bar{j}\rangle\perp}\approx{}^{\mathrm{T}}\mathcal{R}_{\perp\overline{ijk}}-{}^{\mathrm{T}}\mathcal{R}_{\overline{ijk}\perp}\approx 0,\tag{B.29a}$$

$$\bar{\alpha}_2\neq 0\Rightarrow{}^{\mathrm{P}}\mathcal{R}_{\perp\circ}+{}^{\mathrm{P}}\mathcal{R}_{\circ\perp}\approx\underline{\mathcal{R}}_{[\bar{i}\bar{j}]}-\mathcal{R}_{\perp[\bar{i}\bar{j}]\perp}\approx{}^{\mathrm{T}}\mathcal{R}_{\perp\overline{ijk}}+{}^{\mathrm{T}}\mathcal{R}_{\overline{ijk}\perp}\approx 0,\tag{B.29b}$$

$$\bar{\alpha}_3\neq 0\Rightarrow{}^{\mathrm{P}}\mathcal{R}_{\perp\circ}-{}^{\mathrm{P}}\mathcal{R}_{\circ\perp}\approx 0,\tag{B.29c}$$

$$\bar{\alpha}_4\neq 0\Rightarrow\underline{\mathcal{R}}-2\mathcal{R}_{\perp\perp}\approx\mathcal{R}_{\perp\bar{i}}+\mathcal{R}_{\bar{i}\perp}\approx\underline{\mathcal{R}}_{\langle\bar{i}\bar{j}\rangle}-\mathcal{R}_{\perp\langle\bar{i}\bar{j}\rangle\perp}\approx 0,\tag{B.29d}$$

$$\bar{\alpha}_5\neq 0\Rightarrow\underline{\mathcal{R}}_{[\bar{i}\bar{j}]}+\mathcal{R}_{\perp[\bar{i}\bar{j}]\perp}\approx\mathcal{R}_{\perp\bar{i}}-\mathcal{R}_{\bar{i}\perp}\approx 0,\tag{B.29e}$$

$$\bar{\alpha}_6\neq 0\Rightarrow\underline{\mathcal{R}}+2\mathcal{R}_{\perp\perp}\approx 0\tag{B.29f}$$

$$\bar{\beta}_1\neq 0\Rightarrow\mathcal{T}_{\perp\bar{i}\bar{j}}-\mathcal{T}_{[\bar{i}\bar{j}]\perp}\approx\vec{\mathcal{T}}_{\bar{i}}-2\mathcal{T}_{\perp\bar{i}\perp}\approx{}^{\mathrm{T}}\mathcal{T}_{\overline{ijk}}\approx\mathcal{T}_{\langle\bar{i}\bar{j}\rangle\perp}\approx 0,\tag{B.29g}$$

$$\bar{\beta}_2\neq 0\Rightarrow\vec{\mathcal{T}}_{\bar{i}}+\mathcal{T}_{\perp\bar{i}\perp}\approx\mathcal{T}^{\bar{k}}{}_{\bar{k}\perp}\approx 0,\tag{B.29h}$$

$$\bar{\beta}_3\neq 0\Rightarrow\mathcal{T}_{\perp\bar{i}\bar{j}}+2\mathcal{T}_{[\bar{i}\bar{j}]\perp}\approx{}^{\mathrm{P}}\mathcal{T}\approx 0.\tag{B.29i}$$

These final relations are a convenient reference for Section 5.2.3.

Appendix C

Canonical analysis

C.1 Irreducible decomposition of the fields

It is necessary to construct a complete set of idempotent and orthogonal projection operators for the irreducible parts of the field strengths. For general tensors, this can be done with the appropriate $SO^+(1,3)$ Young tableaux, following the methods of [262]. The three projections of the torsion are

$$ {}^{1}\hat{\mathcal{P}}_{ijk}{}^{mnq}\mathcal{T}_{mnq} \equiv \frac{2}{3}\mathcal{T}_{ijk} + \frac{2}{3}\mathcal{T}_{[j|i|k]} + \frac{2}{3}\eta_{i[j}\mathcal{T}_{k]} , \quad \text{(C.1a)} $$

$$ {}^{2}\hat{\mathcal{P}}_{ijk}{}^{mnq}\mathcal{T}_{mnq} \equiv -\frac{2}{3}\eta_{i[j}\mathcal{T}_{k]} , \quad \text{(C.1b)} $$

$$ {}^{3}\hat{\mathcal{P}}_{ijk}{}^{mnq}\mathcal{T}_{mnq} \equiv \frac{1}{6}\epsilon_{ijkl}\epsilon^{lmnq}\mathcal{T}_{mnq} . \quad \text{(C.1c)} $$

The six projections of the Riemann–Cartan curvature are

$$ {}^{1}\hat{\mathcal{P}}_{ijkl}{}^{mnqp}\mathcal{R}_{mnqp} \equiv \frac{1}{3}\mathcal{R}_{ijkl} + \frac{1}{3}\mathcal{R}_{klij} + \frac{2}{3}\mathcal{R}_{[i||[k||j]|l]} - \eta_{[i|[k||}\mathcal{R}_{|j]||l]} - \eta_{[i|[k||}\mathcal{R}_{|l]||j]} + \frac{1}{3}\eta_{i[k|}\eta_{j|l]}\mathcal{R}, \quad \text{(C.2a)} $$

$$ {}^{2}\hat{\mathcal{P}}_{ijkl}{}^{mnqp}\mathcal{R}_{mnqp} \equiv \frac{1}{2}\mathcal{R}_{ijkl} - \frac{1}{2}\mathcal{R}_{klij} - \eta_{[i|[k||}\mathcal{R}_{|j]||l]} + \eta_{[i|[k||}\mathcal{R}_{|l]||j]}, \quad \text{(C.2b)} $$

$$ {}^{3}\hat{\mathcal{P}}_{ijkl}{}^{mnqp}\mathcal{R}_{mnqp} \equiv -\frac{1}{24}\epsilon_{ijkl}\epsilon^{mnop}\mathcal{R}_{mnop}, \quad \text{(C.2c)} $$

$$ {}^{4}\hat{\mathcal{P}}_{ijkl}{}^{mnqp}\mathcal{R}_{mnqp} \equiv \eta_{[i|[k||}\mathcal{R}_{|j]||l]} + \eta_{[i|[k||}\mathcal{R}_{|l]||j]} - \frac{1}{2}\eta_{i[k|}\eta_{j|l]}\mathcal{R}, \quad \text{(C.2d)} $$

$$ {}^{5}\hat{\mathcal{P}}_{ijkl}{}^{mnqp}\mathcal{R}_{mnqp} \equiv \eta_{[i|[k||}\mathcal{R}_{|j]||l]} - \eta_{[i|[k||}\mathcal{R}_{|l]||j]}, \quad \text{(C.2e)} $$

$$ {}^{6}\hat{\mathcal{P}}_{ijkl}{}^{mnqp}\mathcal{R}_{mnqp} \equiv \frac{1}{6}\eta_{i[k|}\eta_{j|l]}\mathcal{R}. \quad \text{(C.2f)} $$

For a translation of the irreducible couplings in Eq. (4.4) into their various alternatives, the reader is directed to Eqs. (B.23j) to (B.23l).

Within Chapter 5 we refer heavily to contractionless tensor quantities with a reduced number of indices, which encode the $SO^+(1,3)$ irreps. For the Riemann–Cartan and torsion tensors, these quantities follow the conventions of [263], with the expansions

$$\mathcal{R}_{ijkl} = \frac{2}{3}\big(2\,{}^{1}\mathcal{R}_{ijkl} + {}^{1}\mathcal{R}_{ikjl}\big) + {}^{2}\mathcal{R}_{ijkl} + {}^{3}\mathcal{R}_{ijkl} + \frac{1}{2}\big(\eta_{ik}({}^{4}\mathcal{R}_{jl} + {}^{5}\mathcal{R}_{jl}) + \eta_{jl}({}^{4}\mathcal{R}_{ik} + {}^{5}\mathcal{R}_{ik})$$
$$- \eta_{jk}({}^{4}\mathcal{R}_{il} + {}^{5}\mathcal{R}_{il}) - \eta_{il}({}^{4}\mathcal{R}_{jk} + {}^{5}\mathcal{R}_{jk})\big) - \frac{1}{12}{}^{6}\mathcal{R}, \tag{C.3a}$$

$$\mathcal{T}_{ijk} = \frac{4}{3}{}^{1}\mathcal{T}_{i[jk]} + \frac{2}{3}\eta_{i[j}\,{}^{2}\mathcal{T}_{j]} + \epsilon_{ijkl}\,{}^{3}\mathcal{T}^{l}. \tag{C.3b}$$

In order to discover the definitions of the numbered irrep tensors in detail, the $SO^{+}(1,3)$ projections can be applied to Eqs. (C.3a) and (C.3b), and the results compared to the definitions in Eqs. (C.1a) to (C.2f). An identical notation is used for the decomposition of the multipliers, and also for the spin tensor density $\sigma^{\mu}{}_{ij}$.

C.2 Ghosts, ranks and signatures

In this appendix, we attempt to elaborate on the motivation of the 'positive kinetic energy test' to which we referred in Section 4.3.1, and which we understand to have been tacitly employed in the previous Hamiltonian treatment of Poincaré gauge theories [169].

Consider the free, vector U(1) theory on $\check{\mathcal{M}}$, without any coupling to gravity (and with Cartesian coordinates $\gamma_{\mu\nu} \equiv \eta_{\mu\nu}$), fixed to the Feynman gauge

$$L_{\mathrm{T}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial_{\mu}A^{\mu})^{2}, \tag{C.4}$$

where we have $F_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]}$. Up to a surface term, (C.4) is equivalent to

$$L_{\mathrm{T}} = -\frac{1}{2}\partial_{\mu}A_{\nu}\partial^{\mu}A^{\nu}, \tag{C.5}$$

which safely propagates four massless polarisations, without developing any classical instability

$$\Box A_{\mu} \approx 0. \tag{C.6}$$

Notwithstanding this reasonable behaviour, we see that the Hamiltonian of (C.5) is unbounded from below

$$\mathcal{H}_{\mathrm{T}} = -\frac{1}{2}\pi_{\mu}\pi^{\mu} + \frac{1}{2}\partial_{\alpha}A_{\mu}\partial^{\alpha}A^{\mu}, \tag{C.7}$$

where the momentum is $\pi_{\mu} \equiv -\partial_{0}A_{\mu}$, since the independent timelike polarisation will have a strictly *negative* contribution. This is naturally revealed in the $3+1$ picture, which we construct by defining a constant unit timelike normal $n^{\mu}n_{\mu} \equiv 1$, and (extending our previous overbar notation to holonomic indices) decomposing quantities into the 0^{+} and 1^{-} irreps

$$A_{\mu} \equiv A_{\perp}n_{\mu} + A_{\overline{\mu}}, \quad \pi_{\mu} \equiv \pi_{\perp}n_{\mu} + \pi_{\overline{\mu}}. \tag{C.8}$$

The Hamiltonian then separates into

$$\mathcal{H}_{\mathrm{T}} = -\frac{1}{2}\pi_{\perp}{}^{2} + \frac{1}{2}\partial_{\alpha}A_{\perp}\partial^{\alpha}A_{\perp} - \frac{1}{2}\pi_{\overline{\mu}}\pi^{\overline{\mu}} + \frac{1}{2}\partial_{\alpha}A_{\overline{\mu}}\partial^{\alpha}A^{\overline{\mu}}, \tag{C.9}$$

where the first and last pairs of terms are respectively negative and positive-definite on the null shell defined by (C.6). The physical consequence is a loss of unitarity: the timelike states have negative norm. In the U(1) theory, this is usually fixed by imposing a Gupta–Bleuler condition on the physical states, which is acceptable since the gauge-fixing term in (C.4) was added by hand anyway. However, in the theories of gravity under consideration, the validity of a Gupta–Bleuler condition is not certain. We note that in the kinetic Hamiltonia of Eqs. (4.20), (4.30), (4.35), (4.40), (4.45), (4.51) and (4.57), we encounter mixed quadratic forms in the momenta, just as we do with the first and third terms of (C.9). If such terms are negative-definite and propagating, we tentatively identify them with a loss of unitarity. We note that without full knowledge of both the nonlinear shell and the remaining field parts of the Hamiltonian (c.f. second and fourth terms in (C.9)), this is quite dangerous. Moreover, as is evident from (C.6), such negative-energy sectors do not necessarily correspond to classical ghosts.

We also mention that the sign of quadratic momenta in the $3+1$ formulation is robust against the choice of signature (as indeed it should be). Recall that throughout this thesis we have used the 'West Coast' signature $(+,-,-,-)$. The sign of each such term may then be inferred by the tensor rank of the momentum irrep, since every contraction on parallel indices introduces a factor of -1. Had we chosen the 'East Coast' signature $(-,+,+,+)$, these factors would not arise. Instead, we would have $n^\mu n_\mu \equiv -1$, whose powers would conspire in the SO(3) decomposition of momenta to have the same effect up to an overall sign in the kinetic Hamiltonian. This final sign is changed by hand in the kinetic part of the Lagrangian, as is customary when changing signature.

C.3 Nonlinear Poisson brackets

In Eqs. (C.15a) to (4.21d) we provide the nonlinear commutators of Case $^{*6}26$. In this appendix we list the emergent commutators of the other theories considered throughout Chapters 4 and 5.

Case 28 The commutators of Case 28 read

$$\left\{\varphi_{\perp\bar{i}},\varphi_{\perp\bar{l}}\right\}\approx \text{RHS of (C.15a)}, \tag{C.10a}$$

$$\left\{\varphi_{\perp\bar{i}},\varphi_{\perp}\right\}\approx -\frac{1}{J^2}\vec{\hat{\pi}}_{\bar{i}}\delta^3, \tag{C.10b}$$

$$\left\{\varphi_{\perp\bar{i}},\widetilde{\varphi}_{\perp\overline{lm}}\right\}\approx \frac{1}{2J^2}\eta_{\bar{i}\langle\bar{l}}\vec{\hat{\pi}}_{\overline{m}\rangle}\delta^3, \tag{C.10c}$$

$$\left\{\varphi_{\perp\bar{i}},{}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\}\approx \frac{1}{2J^2}\left[\eta_{\overline{in}}\overset{\wedge}{\hat{\pi}}_{\perp\overline{lm}}-\frac{1}{2}\eta_{\bar{i}[\bar{l}|}\overset{\wedge}{\hat{\pi}}_{\perp|\overline{m}]\overline{n}}-\frac{3}{4}\eta_{[\bar{l}|\overline{n}}\overset{\wedge}{\hat{\pi}}_{\perp\bar{i}|\overline{m}]}\right]\delta^3, \tag{C.10d}$$

$$\left\{\widetilde{\varphi}_{\overline{ij}},\widetilde{\varphi}_{\overline{lm}}\right\}\approx \text{RHS of (4.21c)}, \tag{C.10e}$$

$$\left\{\widetilde{\varphi}_{\overline{ij}},\widetilde{\varphi}_{\perp\overline{lm}}\right\}\approx \text{RHS of (4.21c)}, \tag{C.10f}$$

$$\begin{aligned}\left\{\widetilde{\varphi}_{\overline{ij}},{}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\}\approx \frac{1}{J^2}\bigg[&\frac{1}{12}\epsilon_{\langle\bar{i}|[\bar{l}\|\overline{n}\perp}\eta_{|\bar{j}\rangle\|\overline{m}]}{}^{\mathrm{P}}\hat{\pi}+\frac{1}{12}\epsilon_{\langle\bar{i}|\overline{lm}\perp}\eta_{|\bar{j}\rangle\overline{n}}{}^{\mathrm{P}}\hat{\pi}+\frac{3}{8}\eta_{\langle\bar{i}|[\bar{l}}\eta_{\overline{m}]\overline{n}}\vec{\hat{\pi}}_{|\bar{j}\rangle}\\ &-\frac{3}{4}\eta_{\langle\bar{i}|\overline{n}}\eta_{|\bar{j}\rangle[\bar{l}}\vec{\hat{\pi}}_{\overline{m}]}\bigg]\delta^3.\end{aligned} \tag{C.10g}$$

In the RHS of (C.10g), we see that the linearly propagating ${}^{\mathrm{P}}\hat{\pi}$ appears, signalling a definite change in the constraint structure when passing from linear to nonlinear regimes.

Case ^{*5}25 The nonlinear commutators of Case ^{*5}25 have been encountered before

$$\left\{\overset{\sim}{\varphi}_{\overline{ij}}, \overset{\sim}{\varphi}_{\overline{lm}}\right\} \approx \text{RHS of (4.21c)}, \tag{C.11a}$$

$$\left\{\overset{\sim}{\varphi}_{\overline{ij}}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\} \approx \text{RHS of (4.21d)}. \tag{C.11b}$$

Again we see that at least (C.11b) is expected to persist on the final shell.

Case 24 Similarly for Case 24 we find

$$\left\{\overset{\sim}{\varphi}_{\overline{ij}}, \overset{\sim}{\varphi}_{\overline{lm}}\right\} \approx \text{RHS of (4.21c)}, \tag{C.12a}$$

$$\left\{\overset{\sim}{\varphi}_{\overline{ij}}, \overset{\sim}{\varphi}_{\perp\overline{lm}}\right\} \approx \frac{1}{J^2}\eta_{(\bar{i}\|(\bar{l}|}\overset{\wedge}{\hat{\pi}}_{\|\bar{j})|\overline{m})}\delta^3, \tag{C.12b}$$

$$\left\{\overset{\sim}{\varphi}_{\overline{ij}}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\} \approx \text{RHS of (C.10g)}. \tag{C.12c}$$

Again we see that at least (C.12c) is expected to persist on the final shell.

Case 3 The nonlinear commutators of Case 3 are all new

$$\left\{\varphi, \overset{\wedge}{\varphi}_{\overline{lm}}\right\} \approx \frac{1}{J^2}\overset{\wedge}{\hat{\pi}}_{\overline{lm}}\delta^3, \tag{C.13a}$$

$$\left\{\overset{\wedge}{\varphi}_{\overline{ij}}, \varphi_{\perp}\right\} \approx -\frac{1}{J^2}\overset{\wedge}{\hat{\pi}}_{\perp\overline{ij}}\delta^3, \tag{C.13b}$$

$$\left\{\overset{\wedge}{\varphi}_{\overline{ij}}, \overset{\sim}{\varphi}_{\perp\overline{lm}}\right\} \approx \frac{1}{J^2}\eta_{[\bar{i}\|\langle\bar{l}|}\overset{\wedge}{\hat{\pi}}_{\perp\|\bar{j}]|\overline{m}\rangle}\delta^3, \tag{C.13c}$$

$$\begin{aligned}\left\{\overset{\wedge}{\varphi}_{\overline{ij}}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\} \approx \frac{1}{J^2}\Bigg[&\frac{1}{12}\epsilon_{[\bar{i}|[\bar{l}\|\overline{n}\perp}\eta_{|\bar{j}]\|\overline{m}]}{}^{\mathrm{P}}\hat{\pi} + \frac{1}{12}\epsilon_{[\bar{i}|\overline{lm}\perp}\eta_{|\bar{j}]\overline{n}}{}^{\mathrm{P}}\hat{\pi} - \frac{1}{8}\epsilon_{\overline{ij}[\bar{l}\perp}\eta_{\overline{m}]\overline{n}}{}^{\mathrm{P}}\hat{\pi} \\ &- \frac{3}{8}\eta_{[\bar{i}|[\bar{l}}\eta_{\overline{m}]\overline{n}}\overset{\rightharpoonup}{\hat{\pi}}_{|\bar{j}]} - \frac{1}{4}\eta_{[\bar{i}|\overline{n}}\eta_{|\bar{j}][\bar{l}}\overset{\rightharpoonup}{\hat{\pi}}_{\overline{m}]} + \frac{1}{4}\eta_{\bar{i}[\bar{l}}\eta_{\overline{m}]\bar{j}}\overset{\rightharpoonup}{\hat{\pi}}_{\overline{n}}\Bigg]\delta^3. \end{aligned} \tag{C.13d}$$

Since (C.13d) also depends on ${}^{\mathrm{P}}\hat{\pi}$, we believe that it will also persist on the final shell. Note that (C.13d) is also linear in $\overset{\rightharpoonup}{\hat{\pi}}_{\overline{k}}$, which we suspect will contribute the massless modes in the linear theory.

Case 17 The nonlinear commutators of Case 17 are of course mostly the same as Case 3

$$\left\{\varphi, \overset{\wedge}{\varphi}_{\overline{lm}}\right\} \approx \text{RHS of (C.13a)}, \tag{C.14a}$$

$$\left\{\overset{\wedge}{\varphi}_{\overline{ij}}, \varphi_{\perp}\right\} \approx \text{RHS of (C.13b)}, \tag{C.14b}$$

$$\left\{\overset{\wedge}{\varphi}_{\overline{ij}}, {}^{\mathrm{P}}\varphi\right\} \approx -\frac{1}{J^2}\eta^{\overline{kl}}\epsilon_{\overline{ijk}\perp}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{l}}\delta^3, \tag{C.14c}$$

$$\left\{\overset{\wedge}{\varphi}_{\overline{ij}}, \overset{\sim}{\varphi}_{\perp\overline{lm}}\right\} \approx \text{RHS of (C.13c)}, \tag{C.14d}$$

$$\left\{\overset{\wedge}{\varphi}_{\overline{ij}}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\} \approx -\frac{1}{J^2}\left[\frac{3}{8}\eta_{[\bar{i}|[\bar{l}}\eta_{\overline{m}]\overline{n}}\overset{\rightharpoonup}{\hat{\pi}}_{|\bar{j}]} + \frac{1}{4}\eta_{[\bar{i}|\overline{n}}\eta_{|\bar{j}][\bar{l}}\overset{\rightharpoonup}{\hat{\pi}}_{\overline{m}]} - \frac{1}{4}\eta_{\bar{i}[\bar{l}}\eta_{\overline{m}]\bar{j}}\overset{\rightharpoonup}{\hat{\pi}}_{\overline{n}}\right]\delta^3. \tag{C.14e}$$

Note that (C.14e) is linear in $\overset{\rightharpoonup}{\hat{\pi}}_{\overline{k}}$, the momentum of the tentative ‘vector’ graviton.

Case 16 The nonlinear commutators of Case 16 – as considered in Chapter 5 – are structurally different from those appearing in the theories of Chapter 4. The appearance in (5.4) of PiCs which depend on the Riemann–Cartan curvature leads to commutators which are simple linear combinations of the momenta and the field strengths

$$\left\{\hat{\varphi}_{\bar{i}\bar{j}}, \varphi_{\perp}\right\} \approx -\frac{1}{J}\left(\frac{1}{J}\hat{\pi}_{\perp\bar{i}\bar{j}} + 8\hat{\alpha}_6 \underline{\mathcal{R}}_{[\bar{i}\bar{j}]}\right)\delta^3, \tag{C.15a}$$

$$\begin{aligned}\left\{\hat{\varphi}_{\bar{i}\bar{j}}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\} \approx{}& \frac{1}{6J}{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{lmn}}{}^{\overline{pqr}}\Big[\eta_{\bar{r}[\bar{i}}\epsilon_{\bar{j}]\overline{pq}\perp}\left(\frac{1}{J}{}^{\mathrm{P}}\hat{\pi} - 16\hat{\alpha}_6{}^{\mathrm{P}}\mathcal{R}_{\perp\circ}\right) \\ &+3\eta_{[\bar{p}|[\bar{i}}\eta_{\bar{j}]|\bar{q}]}\left(\frac{1}{J}\vec{\hat{\pi}}_{\bar{r}} + 16\hat{\alpha}_6\mathcal{R}_{\perp\bar{r}}\right) - 256\hat{\alpha}_6\eta_{[\bar{p}|[\bar{i}}{}^{\mathrm{T}}\mathcal{R}_{\perp\bar{j}]|\bar{q}]\bar{r}} + 128\hat{\alpha}_6\eta_{\bar{r}[\bar{i}|}{}^{\mathrm{T}}\mathcal{R}_{\perp\overline{pq}|\bar{j}]}\Big]\delta^3,\end{aligned} \tag{C.15b}$$

$$\left\{\tilde{\varphi}_{\bar{i}\bar{j}}, \varphi_{\perp}\right\} \approx -\frac{1}{J}\left(\frac{1}{J}\tilde{\hat{\pi}}_{\perp\bar{i}\bar{j}} + 8\hat{\alpha}_6 \underline{\mathcal{R}}_{\langle\bar{i}\bar{j}\rangle}\right)\delta^3, \tag{C.15c}$$

$$\begin{aligned}\left\{\tilde{\varphi}_{\bar{i}\bar{j}}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\} \approx{}& \frac{1}{6J}{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{lmn}}{}^{\overline{pqr}}\Big[\eta_{\bar{r}\langle\bar{i}}\epsilon_{\bar{j}\rangle\overline{pq}\perp}\left(\frac{1}{J}{}^{\mathrm{P}}\hat{\pi} + 16\hat{\alpha}_6{}^{\mathrm{P}}\mathcal{R}_{\perp\circ}\right) \\ &-6\eta_{\bar{r}\langle\bar{i}}\eta_{\bar{j}\rangle[\bar{p}}\left(\frac{1}{J}\vec{\hat{\pi}}_{\bar{q}]} + 16\hat{\alpha}_6\mathcal{R}_{\perp\bar{q}]}\right) + 256\hat{\alpha}_6\eta_{[\bar{p}|\langle\bar{i}}{}^{\mathrm{T}}\mathcal{R}_{\perp\bar{j}\rangle|\bar{q}]\bar{r}} + 128\hat{\alpha}_6\eta_{\bar{r}\langle\bar{i}|}{}^{\mathrm{T}}\mathcal{R}_{\perp\overline{pq}|\bar{j}\rangle}\Big]\delta^3,\end{aligned} \tag{C.15d}$$

$$\left\{\varphi_{\perp}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\} \approx -\frac{24\hat{\alpha}_6}{J}{}^{\mathrm{T}}\mathcal{T}_{\overline{lmn}}\,\delta^3, \tag{C.15e}$$

$$\left\{{}^{\mathrm{T}}\varphi_{\overline{ijk}}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\right\} \approx \frac{64\hat{\alpha}_6}{J}{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{ijk}}{}^{\overline{pqr}}\,{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{lmn}}{}^{\overline{uvw}}\Big[\eta_{\overline{rw}}\eta_{[\bar{p}|[\bar{u}}\mathcal{T}_{\perp\bar{v}]|\bar{q}]} - \eta_{\bar{w}[\bar{p}}\eta_{\bar{p}][\bar{u}}\mathcal{T}_{\perp\bar{v}]\bar{r}}\Big]\delta^3. \tag{C.15f}$$

Case 16 with tensor bypass The action of the torsion multiplier irrep ${}^{1}\lambda^{i}{}_{jk}$ on Case 16 has a powerful effect on the commutators of its PiCs, while also adding new primary and secondary constraints. The augmented PPM of the resulting theory contains the following commutators not listed already in Eqs. (C.15a) to (C.15e)

$$\left\{\hat{\varphi}_{\bar{i}\bar{j}}, \hat{\varphi}_{\bar{l}\bar{m}}\right\} \approx \frac{4\bar{\beta}_1}{3J}m_{\mathrm{p}}{}^2\eta_{[\bar{i}\|[\bar{l}|}\left(\lambda_{\perp|\bar{m}]\|\bar{j}]} - \lambda_{[|\bar{m}]\|\bar{j}]]\perp}\right)\delta^3, \tag{C.16a}$$

$$\left\{\hat{\varphi}_{\bar{i}\bar{j}}, \tilde{\varphi}_{\bar{l}\bar{m}}\right\} \approx \frac{4\bar{\beta}_1}{J}m_{\mathrm{p}}{}^2\eta_{[\bar{i}\|[\bar{l}|}\lambda_{\langle|\bar{m}\|\bar{j}]\rangle\perp}\delta^3, \tag{C.16b}$$

$$\begin{aligned}\left\{\hat{\varphi}_{\bar{i}\bar{j}}, {}^{\mathrm{T}}\underline{\chi}^{\parallel}_{\overline{lmn}}\right\} \approx{}& \Bigg[{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{lmnij}}{}^{\bar{k}}\left[\mathcal{D}_{\bar{k}}\left(\frac{1}{J}\right) + \frac{1}{J}\vec{\mathcal{T}}_{\bar{k}}\right] + \frac{1}{4J}\eta_{[\bar{i}\|[\bar{l}|}n_n\mathcal{D}_{|\bar{m}]}n_{\|j]} - \frac{1}{4J}\eta_{[\bar{i}\|[\bar{l}|}n_{|m]}\mathcal{D}_{\bar{n}}n_{\|j]} \\ &+\frac{1}{2J}\eta_{\bar{n}[\bar{i}|}n_{[l\|}\mathcal{D}_{|\bar{j}]}n_{\|m]} + \frac{1}{2J}\eta_{\bar{n}[\bar{i}}\eta_{\bar{j}][\bar{l}}n_{m]}\eta^{\overline{pq}}\mathcal{D}_{\bar{p}}n_q - \frac{1}{2J}\eta_{\bar{l}[\bar{i}}\eta_{\bar{j}]\bar{m}}n_n\eta^{\overline{pq}}\mathcal{D}_{\bar{p}}n_q \\ &-\frac{3}{4J}\eta_{[\bar{l}\|[\bar{i}|}n_n\mathcal{T}_{\perp|\bar{j}]\|\bar{m}]} - \frac{3}{4J}\eta_{[\bar{l}\|[\bar{i}|}n_{\|m]}\mathcal{T}_{\perp|\bar{j}]\bar{n}} - \frac{1}{2J}\eta_{\bar{n}[\bar{i}}\mathcal{T}_{\perp\bar{j}][\bar{l}}n_{m]} \\ &-\frac{3}{8J}\eta_{\bar{n}[\bar{l}}n_{m]}\mathcal{T}_{\perp\bar{i}\bar{j}}\Bigg]\delta^3 + \frac{1}{J}{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{lmnij}}{}^{\bar{k}}\delta^3\mathcal{D}_{\bar{k}},\end{aligned} \tag{C.16c}$$

$$\begin{aligned}\left\{\hat{\varphi}_{\bar{i}\bar{j}}, \chi^{\perp\!\!\perp}_{\perp\bar{l}}\right\} \approx{}& m_{\mathrm{p}}{}^2\Bigg[\eta_{\bar{l}[\bar{i}|}\left(2\hat{\beta}_2\mathcal{D}_{|\bar{j}]}\left(\frac{1}{J}\right) - \frac{\bar{\beta}_1}{J}\vec{\lambda}_{|\bar{j}]} + \frac{2\bar{\beta}_1}{J}\lambda_{\perp|\bar{j}]\perp}\right) + \frac{4\bar{\beta}_1}{3J}{}^{\mathrm{T}}\lambda_{\overline{ijl}} \\ &-\frac{\hat{\beta}_2}{J}\mathcal{T}_{\perp\bar{i}\bar{j}}\,n_l\Bigg]\delta^3 + \frac{2\hat{\beta}_2}{J}m_{\mathrm{p}}{}^2\delta^3\eta_{\bar{l}[\bar{i}}\mathcal{D}_{\bar{j}]},\end{aligned} \tag{C.16d}$$

$$\left\{\tilde{\varphi}_{\bar{i}\bar{j}}, \tilde{\varphi}_{\bar{l}\bar{m}}\right\} \approx \frac{4\bar{\beta}_1}{3J}m_{\mathrm{p}}{}^2\eta_{(\bar{i}\|(\bar{l}|}\left(\lambda_{\perp|\bar{m})\|\bar{j})} - \lambda_{[|\bar{m}\|)\bar{j})]\perp}\right)\delta^3, \tag{C.16e}$$

$$
\begin{aligned}
\Big\{\tilde{\varphi}_{\overline{ij}}, {}^{\mathrm{T}}\underline{\chi}^{\parallel}_{\overline{lmn}}\Big\} \approx \Bigg[& 2\,{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{lmn}}{}^{\overline{k}}{}_{\langle\overline{ij}\rangle}\left[\mathcal{D}_{\overline{k}}\Big(\frac{1}{J}\Big)+\frac{1}{J}\overrightarrow{\mathcal{T}}_{\overline{k}}\right]-\frac{2}{3J}\,{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{lmn}\langle\bar{i}|}{}^{\overline{pq}}\epsilon_{\overline{pq}|\bar{j}\rangle\perp}{}^{\mathrm{P}}\mathcal{T} \\
&+\frac{3}{4J}\eta_{\langle\bar{i}\|[\bar{l}|}n_{n}\mathcal{D}_{|\overline{m}]}n_{\|j\rangle}-\frac{3}{4J}\eta_{\langle\bar{i}\|[\bar{l}|}n_{|m]}\mathcal{D}_{\overline{n}}n_{\|j\rangle}+\frac{3}{2J}\eta_{\overline{n}\langle\bar{i}\|}n_{[l|}\mathcal{D}_{\|\bar{j}\rangle}n_{|m]} \\
&+\frac{3}{4J}\eta_{\overline{n}[\bar{l}}n_{m]}\mathcal{D}_{\langle\bar{i}}n_{j\rangle}+\frac{3}{2J}\eta_{\overline{n}\langle\bar{i}}\eta_{j\rangle[\bar{l}}n_{m]}\eta^{\overline{pq}}\mathcal{D}_{\overline{p}}n_{q}-\frac{3}{4J}\eta_{\langle\bar{i}\|[\bar{l}|}n_{n}\mathcal{T}_{\perp|\overline{m}]\|\bar{j}\rangle} \\
&-\frac{3}{4J}\eta_{\langle\bar{i}\|[\bar{l}|}n_{|m]}\mathcal{T}_{\perp\overline{n}\|\bar{j}\rangle}+\frac{3}{2J}\eta_{\overline{n}\langle\bar{i}\|}n_{[l|}\mathcal{T}_{\perp\|\bar{j}\rangle|\overline{m}]}\Bigg]\delta^{3}+\frac{2}{J}\,{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{lmn}}{}^{\overline{k}}{}_{\langle\overline{ij}\rangle}\delta^{3}\mathcal{D}_{\overline{k}},
\end{aligned}
\tag{C.16f}
$$

$$
\begin{aligned}
\Big\{\tilde{\varphi}_{\overline{ij}}, \chi^{\perp\!\!\!\perp}_{\perp\bar{l}}\Big\} \approx {m_{\mathrm{p}}}^{2}\Bigg[&\eta_{\bar{l}\langle\bar{i}|}\left(2\hat{\beta}_{2}\mathcal{D}_{|\bar{j}\rangle}\Big(\frac{1}{J}\Big)-\frac{\bar{\beta}_{1}}{J}\overrightarrow{\lambda}_{|\bar{j}\rangle}+\frac{2\bar{\beta}_{1}}{J}\lambda_{\perp|\bar{j}\rangle\perp}\right)-\frac{8\bar{\beta}_{1}}{3J}\,{}^{\mathrm{T}}\lambda_{\langle\bar{i}|\bar{l}|\bar{j}\rangle} \\
&+\frac{\hat{\beta}_{2}}{J}n_{l}\mathcal{D}_{\langle\bar{i}}n_{j\rangle}\Bigg]\delta^{3}-\frac{2\hat{\beta}_{2}}{J}{m_{\mathrm{p}}}^{2}\delta^{3}\eta_{\bar{l}\langle\bar{i}}\mathcal{D}_{\bar{j}\rangle},
\end{aligned}
\tag{C.16g}
$$

$$
\Big\{\varphi_{\perp}, \chi^{\perp\!\!\!\perp}_{\perp\bar{l}}\Big\} \approx \frac{1}{J}\Big(\overrightarrow{\hat{\pi}}_{\bar{l}}-8\hat{\alpha}_{6}\mathcal{R}_{\perp\bar{l}}\Big)\delta^{3},
\tag{C.16h}
$$

$$
\Big\{{}^{\mathrm{T}}\varphi_{\overline{ijk}}, {}^{\mathrm{T}}\underline{\chi}^{\parallel}_{\overline{lmn}}\Big\} \approx -\frac{2}{J}\,{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{ijk}}{}^{\overline{pqr}}\,{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{lmnpqr}}\delta^{3},
\tag{C.16i}
$$

$$
\Big\{{}^{\mathrm{T}}\varphi_{\overline{ijk}}, \chi^{\perp\!\!\!\perp}_{\perp\bar{l}}\Big\} \approx \frac{2}{J}\,{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{ijk}}{}^{\overline{pqr}}\eta_{\bar{l}[\bar{p}|}\Big(\frac{1}{J}\overset{\wedge}{\hat{\pi}}_{\perp|\bar{q}]\bar{r}}+\frac{1}{J}\overset{\sim}{\hat{\pi}}_{\perp|\bar{q}]\bar{r}}-32\hat{\alpha}_{6}\underline{\mathcal{R}}_{[|\bar{q}]\bar{r}]}-32\hat{\alpha}_{6}\underline{\mathcal{R}}_{\langle|\bar{q}]\bar{r}\rangle}\Big)\delta^{3},
\tag{C.16j}
$$

$$
\Big\{{}^{\mathrm{T}}\underline{\chi}^{\parallel}_{\overline{ijk}}, \chi^{\perp\!\!\!\perp}_{\perp\bar{l}}\Big\} \approx -\frac{2}{J}\,{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{ijk}}{}^{\overline{pqr}}\eta_{\bar{l}[\bar{p}|}\Big(\mathcal{T}_{\perp|\bar{q}]\bar{r}}+\mathcal{D}_{|\bar{q}]}n_{r}\Big)\delta^{3},
\tag{C.16k}
$$

$$
\Big\{\chi^{\perp\!\!\!\perp}_{\perp\bar{i}}, \chi^{\perp\!\!\!\perp}_{\perp\bar{l}}\Big\} \approx \left[\frac{2\hat{\beta}_{2}}{J}{m_{\mathrm{p}}}^{2}\mathcal{T}_{\perp\overline{il}}-\frac{8\bar{\beta}_{1}}{3J}{m_{\mathrm{p}}}^{2}\Big(\lambda_{\perp\overline{il}}-\lambda_{[\overline{il}]\perp}\Big)\right]\delta^{3}.
\tag{C.16l}
$$

We particularly note the appearance among Eqs. (C.16a) to (C.16l) of desirable $\mathcal{O}(1)$ commutators.

C.4 Heuristic outlook

In this appendix we attempt to quantify the chances of the catalogue of [153] containing a viable theory, in light of the initial survey performed in Chapter 4. Let k viable theories be found in a sample of $n = 8$, drawn from a population of $N = 58$ theories. We may model the probability of there being a grand total of K viable theories in the parent population as

$$
P(K|k,n,N) \equiv \frac{n+1}{N+1}P_{\mathrm{hyp}}(k|K,n,N),
\tag{C.17}
$$

where the probability $P_{\mathrm{hyp}}(k|K,n,N)$ of drawing k given K follows the standard hypergeometric distribution $P_{\mathrm{hyp}}(k|K,n,N) \equiv \binom{K}{k}\binom{N-K}{n-k}/\binom{N}{n}$. Note that we have assumed a uniform prior on K, $P(K|N) \equiv (N+1)^{-1}$, which may or may not be justified. The pessimistic interpretation of Chapter 4 would be $k = 0$, but in that case the probability that $K = 0$ is found to be only 0.15 according to (C.17). Rather, we would then expect $K = 5 \pm 4.9$. Moreover, the pessimistic interpretation is not necessarily the most conservative, since Case 20 and Case 32 are not ruled out at the level of the PPM: we would expect $K = 11 \pm 6.6$ and $K = 17 \pm 7.6$ for $k = 1$ and $k = 2$ respectively. This outlook is more promising, but still assumes a uniform prior which might be improved by considering the methods used to obtain the cases, from a theoretical perspective. In any case, it is clear that further study of the remaining theories will be necessary to draw firm conclusions, and we begin this project in Chapter 5.

C.5 Constrained vs tensorial degrees of freedom

The constraints encountered within Chapters 4 and 5 are typically not scalars, and so in this appendix we will distinguish between the truly *constrained* D.o.F and the apparent D.o.F implied by their tensor structure. To illustrate how this difference can arise, we extend our discussion in Appendix C.2 to consider another simple theory on $\check{\mathcal{M}}$

$$L_{\mathrm{T}} = \phi\partial_\mu A^\mu + \varepsilon A^\mu \partial_\mu \phi, \tag{C.18}$$

which is a total divergence when the constant parameter $\varepsilon = 1$. We shall assume $\varepsilon \neq 1$, in which case it is still clear from the Euler–Lagrange equations that the theory (C.18) does not propagate any of the five field D.o.F in the scalar ϕ and vector A^μ, since $(1-\varepsilon)\partial_\mu A^\mu \approx (\varepsilon - 1)\partial_\mu \phi \approx 0$, and we can obtain this result formally from the Hamiltonian analysis. None of the field momenta are soluble in terms of the velocities, and so there are five primary constraints $\varphi \equiv \pi - \varepsilon A^0 \approx 0$, $\varphi_0 \equiv \pi_0 - \phi \approx 0$ and $\varphi_\alpha \equiv \pi_\alpha \approx 0$, leaving the total Hamiltonian $\mathcal{H}_{\mathrm{T}} = -\phi\partial_\alpha A^\alpha - \varepsilon A^\alpha \partial_\alpha \phi + u\varphi + u^0\varphi_0 + u^\alpha \varphi_\alpha$. The consistency conditions with $\mathcal{H}_{\mathrm{T}}$ reveal that the two scalar primaries fail to commute with each other and are SC according to $\{\varphi, \mathcal{H}_{\mathrm{T}}\} \approx (1-\varepsilon)(u^0 + \partial_\alpha A^\alpha)$ and $\{\varphi_0, \mathcal{H}_{\mathrm{T}}\} \approx (\varepsilon - 1)u$, while the vector primary suggests a secondary $\{\varphi_\alpha, \mathcal{H}_{\mathrm{T}}\} \approx (\varepsilon - 1)\partial_\alpha \phi$. The secondary $\partial_\alpha \phi \approx 0$ fails to commute with the primary φ, so that the determined multiplier $u \approx 0$ already ensures its consistency without invoking a tertiary constraint: the algorithm is thus terminated. In the final counting we recall that it is necessary to extract all null eigenvectors from the PPM, and indeed the secondary $\partial_\alpha \phi$ can be promoted to the FC combination

$$\chi_\alpha \equiv (1-\varepsilon)\partial_\alpha \phi - \varphi_0 \partial_\alpha \pi \approx 0. \tag{C.19}$$

We are thus left with the FC constraints φ_α and χ_α, and SC constraints φ and φ_0. This yields zero D.o.F as expected

$$0 = \frac{1}{2}\big(10 - 2 \times (3+1)[\mathrm{FC}] - (1+1)[\mathrm{SC}]\big). \tag{C.20}$$

The key observation in (C.20) is that $\varphi_\alpha \approx 0$ constrains the *three* independent momentum D.o.F in π_α – one for each D.o.F in its vector structure – whilst the independent part of the vector $\chi_\alpha \approx 0$ in (C.19) constrains only the *single* field D.o.F in ϕ.

C.6 Linearisation of sure primary first-class constraints

In this appendix we will consider the safety of including the linearised sSFCs in the final D.o.F count. We recall that the Poincaré gauge symmetry implies the existence of 10 sSFCs, labelled $\mathcal{H}_\perp$, $\mathcal{H}_\alpha$, $\mathcal{H}_{\bar{i}\bar{j}}$ and $\mathcal{H}_{\perp\bar{i}}$. However we frequently found in Chapter 4 that some of these quantitites were missing when linearised on the PiC shell. An sSFC may clearly vanish if it is an arbitrary linear combination of iPFCs, consistent with its FC property; how then to interpret an sSFC which happens to be an arbitrary linear combination of iSSCs?

This problem is resolved when we see that Eqs. (4.15a) to (4.15d) are *incomplete* formulae for the sSFCs when iPSCs are present in the theory. Let the super-Hamiltonian be, to lowest perturbative order, a linear combination of the only two iPSCs which appear in a given theory

$$\mathcal{H}^\flat{}_\perp \equiv c_A{}^{\acute{u}A}\varphi^\flat{}_{\acute{u}} + c_E{}^{\acute{u}E}\varphi^\flat{}_{\acute{u}} \approx 0, \tag{C.21}$$

where we employ the notation of Chapter 5, and note that the only nonvanishing commutator between PiCs $\left\{{}^{A}\varphi^{\flat}{}_{\acute{u}},\,{}^{E}\varphi^{\flat}{}_{\acute{u}}\right\}$ will be of order unity. The total Hamiltonian will take the form

$$\mathcal{H}_{\mathrm{T}} \equiv N\mathcal{H}^{\flat}{}_{\perp} + {}^{A}u^{\flat\,\acute{u}}\,{}^{A}\varphi^{\flat}{}_{\acute{u}} + {}^{E}u^{\flat\,\acute{u}}\,{}^{E}\varphi^{\flat}{}_{\acute{u}} + \cdots \equiv N\overline{\mathcal{H}}^{\flat}{}_{\perp} + \cdots, \tag{C.22}$$

where the ellipsis in (C.22) include the remaining sSFCs, iPFCs and surface terms, and all higher-order terms. The *modified* super-Hamiltonian is formed by solving for the PiC multipliers, and we have

$$\overline{\mathcal{H}}^{\flat}{}_{\perp} \equiv \mathcal{H}^{\flat}{}_{\perp} - \left(\left\{{}^{E}\varphi^{\flat},{}^{A}\varphi^{\flat}\right\}^{-1}\right)^{\acute{v}}{}_{\acute{u}}\left\{{}^{E}\varphi^{\flat\,\acute{u}},\mathcal{H}^{\flat}{}_{\perp}\right\}{}^{A}\varphi^{\flat}{}_{\acute{v}} + (A\leftrightarrow E) \approx 0. \tag{C.23}$$

The quantity defined in (C.23) is the linearisation of the *complete* sure secondary, and is FC by construction. Moreover, we can see by substituting from (C.21) that even this complete quantity will vanish, with or without reference to the PiC shell. The argument can be generalised to arbitrarily many iPSCs, and to the remaining sSFCs.

C.7 Incomplete analysis of Case 16

We provide in this appendix a partial analysis of Case 16. The nonlinear sSFCs, evaluated on the PiC shell, are

$$\begin{aligned}
\mathcal{H}_{\perp} \approx \frac{J}{96}\Bigg[&-\frac{1}{(\hat{\alpha}_3+2\hat{\alpha}_6)J}{}^{\mathrm{P}}\hat{\pi}\left(\frac{1}{J}{}^{\mathrm{P}}\hat{\pi}-8(\hat{\alpha}_3-2\hat{\alpha}_6){}^{\mathrm{P}}\mathcal{R}_{\perp\circ}\right)\\
&+\frac{12}{(\hat{\alpha}_5+2\hat{\alpha}_6)J}\overset{\wedge}{\hat{\pi}}{}^{\perp\overline{ij}}\left(\frac{1}{J}\overset{\wedge}{\hat{\pi}}_{\perp\overline{ij}}+8(\hat{\alpha}_5-2\hat{\alpha}_6)\underline{\mathcal{R}}_{[\overline{ij}]}\right)+\frac{6}{(\hat{\alpha}_5-\hat{\alpha}_6)J}\overset{\rightharpoonup}{\hat{\pi}}{}^{\bar{i}}\left(\frac{1}{J}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{i}}-8(\hat{\alpha}_5+\hat{\alpha}_6)\mathcal{R}_{\perp\bar{i}}\right)\\
&-\frac{4}{\hat{\alpha}_6 J}\overset{\sim}{\hat{\pi}}{}^{\perp\overline{ij}}\left(\frac{1}{J}\overset{\sim}{\hat{\pi}}_{\perp\overline{ij}}-8\hat{\alpha}_6\underline{\mathcal{R}}_{\langle\overline{ij}\rangle}\right)+\frac{4}{\hat{\beta}_2 J}\hat{\pi}^2+\frac{36}{\hat{\beta}_2 J}\hat{\pi}^{\perp\bar{i}}\left(\frac{1}{J}\hat{\pi}_{\perp\bar{i}}+\frac{8\hat{\beta}_2}{3}\overset{\rightharpoonup}{\mathcal{T}}_{\bar{i}}\right)+\frac{128\hat{\alpha}_3\hat{\alpha}_6}{\hat{\alpha}_3+2\hat{\alpha}_6}{}^{\mathrm{P}}\mathcal{R}_{\perp\circ}{}^2\\
&-\frac{1536\hat{\alpha}_5\hat{\alpha}_6}{\hat{\alpha}_5+2\hat{\alpha}_6}\underline{\mathcal{R}}^{[\overline{ij}]}\underline{\mathcal{R}}_{[\overline{ij}]}+\frac{384\hat{\alpha}_5\hat{\alpha}_6}{\hat{\alpha}_5-\hat{\alpha}_6}\mathcal{R}^{\perp\bar{i}}\mathcal{R}_{\perp\bar{i}}+512\hat{\alpha}_6\underline{\mathcal{R}}^{\langle\overline{ij}\rangle}\underline{\mathcal{R}}_{\langle\overline{ij}\rangle}\Bigg]+\frac{1}{3}\hat{\pi}\eta^{\overline{kl}}\mathcal{D}_{\bar{k}}n_l\\
&-J\eta^{\overline{kl}}\left[\mathcal{D}_{\bar{k}}\left(\frac{1}{J}\hat{\pi}_{\perp\bar{l}}\right)+\frac{1}{J}\hat{\pi}_{\perp\bar{l}}\overset{\rightharpoonup}{\mathcal{T}}_{\bar{k}}\right]\approx 0,
\end{aligned}\tag{C.24a}$$

$$\begin{aligned}
\mathcal{H}_{\bar{i}} \approx \frac{J}{6}\Bigg[&-8\eta^{\overline{kp}}\eta^{\overline{lq}}\left[\left(\frac{1}{J}\overset{\wedge}{\hat{\pi}}_{\perp\overline{kl}}-16\hat{\alpha}_6\underline{\mathcal{R}}_{[\overline{kl}]}\right)-\left(\frac{1}{J}\overset{\sim}{\hat{\pi}}_{\perp\overline{kl}}-16\hat{\alpha}_6\underline{\mathcal{R}}_{\langle\overline{kl}\rangle}\right)\right]{}^{\mathrm{T}}\mathcal{R}_{\perp\overline{ipq}}\\
&-\left(\frac{1}{J}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{i}}-8\hat{\alpha}_6\mathcal{R}_{\perp\bar{i}}\right)\underline{\mathcal{R}}-3\eta^{\overline{kl}}\left[\left(\frac{1}{J}\overset{\wedge}{\hat{\pi}}_{\perp\overline{ik}}\mathcal{R}_{\perp\bar{l}}+\frac{1}{J}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{l}}\underline{\mathcal{R}}_{[\overline{ik}]}\right)+\left(\frac{1}{J}\overset{\sim}{\hat{\pi}}_{\perp\overline{ik}}\mathcal{R}_{\perp\bar{l}}+\frac{1}{J}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{l}}\underline{\mathcal{R}}_{\langle\overline{ik}\rangle}\right)\right]\\
&-\eta^{\overline{kp}}\eta^{\overline{lq}}\epsilon_{\overline{ikl}\perp}\left(\frac{1}{J}\overset{\wedge}{\hat{\pi}}_{\perp\overline{pq}}{}^{\mathrm{P}}\mathcal{R}_{\perp\circ}+\frac{1}{J}{}^{\mathrm{P}}\hat{\pi}\underline{\mathcal{R}}_{[\overline{pq}]}\right)\Bigg]-\eta^{\overline{kl}}\hat{\pi}_{\perp\bar{k}}\mathcal{D}_{\bar{l}}n_i-\frac{J}{3}\mathcal{D}_{\bar{i}}\left(\frac{1}{J}\hat{\pi}\right)\approx 0,
\end{aligned}\tag{C.24b}$$

$$\begin{aligned}
\mathcal{H}_{\overline{ij}} \approx{}& \frac{4\hat{\alpha}_6 J}{3}\mathcal{T}_{\perp\overline{ij}}\,\underline{\mathcal{R}}-\frac{64\hat{\alpha}_6 J}{3}\eta^{\overline{mn}}\delta^{\bar{k}}_{\bar{i}}\delta^{\bar{l}}_{\bar{j}}\left(\overset{\rightharpoonup}{\mathcal{T}}_{\overline{m}}{}^{\mathrm{T}}\mathcal{R}_{\perp\overline{kln}}+\mathcal{D}_{\overline{m}}{}^{\mathrm{T}}\mathcal{R}_{\perp\overline{kln}}\right)-\eta^{\overline{kl}}\left(\overset{\sim}{\hat{\pi}}_{\perp\bar{k}[\bar{i}|}-\overset{\wedge}{\hat{\pi}}_{\perp\bar{k}[\bar{i}|}\right)\mathcal{D}_{|\bar{j}]}n_l\\
&-J\delta^{\bar{l}}_{[\bar{i}|}\left[\mathcal{D}_{|\bar{j}]}\left(\frac{1}{J}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{l}}\right)+\frac{1}{J}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{l}}\overset{\rightharpoonup}{\mathcal{T}}_{|\bar{j}]}\right]-\frac{J}{6}\eta^{\overline{lk}}\epsilon_{\overline{ijl}\perp}\left[\mathcal{D}_{\bar{k}}\left(\frac{1}{J}{}^{\mathrm{P}}\hat{\pi}\right)+\frac{1}{J}{}^{\mathrm{P}}\hat{\pi}\overset{\rightharpoonup}{\mathcal{T}}_{\bar{k}}\right]\approx 0,
\end{aligned}\tag{C.24c}$$

$$\begin{aligned}
\mathcal{H}_{\perp\bar{i}} \approx{}& \hat{\pi}_{\perp\bar{i}}-\frac{4\hat{\alpha}_6 J}{3}\left(\overset{\rightharpoonup}{\mathcal{T}}_{\bar{i}}\underline{\mathcal{R}}+\mathcal{D}_{\bar{i}}\underline{\mathcal{R}}\right)-\frac{64\hat{\alpha}_6 J}{3}\eta^{\overline{mn}}\eta^{pl}\delta^{\bar{k}}_{\bar{i}}n_p\mathcal{D}_{\overline{m}}{}^{\mathrm{T}}\mathcal{R}_{\perp\overline{kln}}\\
&+\frac{1}{2}\eta^{\overline{kl}}\left(n_k\mathcal{D}_{\bar{i}}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{l}}+\overset{\rightharpoonup}{\hat{\pi}}_{\bar{i}}\mathcal{D}_k n_l\right)-J\eta^{\overline{pq}}\delta^{\bar{l}}_{\bar{i}}\left[\mathcal{D}_{\bar{p}}\left(\frac{1}{J}\overset{\wedge}{\hat{\pi}}_{\perp\overline{lq}}\right)+\frac{1}{J}\overset{\wedge}{\hat{\pi}}_{\perp\overline{ql}}\overset{\rightharpoonup}{\mathcal{T}}_{\bar{p}}\right]
\end{aligned}$$

$$-J\eta^{\overline{pq}}\delta^{\bar{l}}_{\bar{i}}\left[\mathcal{D}_{\overline{p}}\left(\frac{1}{J}\tilde{\hat{\pi}}_{\perp\overline{lq}}\right)+\frac{1}{J}\tilde{\hat{\pi}}_{\perp\overline{ql}}\vec{\mathcal{T}}_{\overline{p}}\right]+\frac{1}{12}\eta^{\overline{kp}}\eta^{\overline{lq}}\epsilon_{ikl\perp}{}^{\mathrm{P}}\hat{\pi}\mathcal{T}_{\perp\overline{pq}}\approx 0. \tag{C.24d}$$

The nonlinear quantities Eqs. (C.24a) to (C.24d) are independent sSFCs, and in their full complexity remove the expected 2×10 canonical D.o.F. In the linear theory these quantities become

$$\begin{aligned}
&\mathcal{H}^{\flat}{}_{\perp}\approx-\eta^{\flat\overline{kl}}\mathcal{D}^{\flat}{}_{\overline{k}}\hat{\pi}^{\flat}{}_{\perp\overline{l}}\approx 0,\quad \mathcal{H}^{\flat}{}_{\overline{i}}\approx-\frac{1}{3}\mathcal{D}^{\flat}{}_{\overline{i}}\hat{\pi}^{\flat}\approx 0,\\
&\mathcal{H}^{\flat}{}_{\overline{ij}}\approx-\frac{1}{6}\eta^{\flat\overline{kl}}\epsilon^{\flat}{}_{\overline{ijk}\perp}\mathcal{D}^{\flat}{}_{\overline{l}}{}^{\mathrm{P}}\hat{\pi}^{\flat}+\mathcal{D}^{\flat}{}_{[\overline{i}}\vec{\hat{\pi}}^{\flat}{}_{\overline{j}]}-\frac{64\hat{\alpha}_6 J^{\flat}}{3}\eta^{\flat\overline{mn}}\delta^{\flat\overline{k}}_{\overline{i}}\delta^{\flat\overline{l}}_{\overline{j}}\mathcal{D}^{\flat}{}_{\overline{m}}{}^{\mathrm{T}}\mathcal{R}^{\flat}{}_{\perp\overline{kln}}\approx 0,\\
&\mathcal{H}^{\flat}{}_{\perp\overline{i}}\approx\hat{\pi}^{\flat}{}_{\perp\overline{i}}-\eta^{\flat\overline{mn}}\mathcal{D}^{\flat}{}_{\overline{m}}\left(\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{in}}+\tilde{\hat{\pi}}^{\flat}{}_{\perp\overline{in}}\right)-\frac{4\hat{\alpha}_6 J^{\flat}}{3}\mathcal{D}^{\flat}{}_{\overline{i}}\underline{\mathcal{R}}^{\flat}\approx 0.
\end{aligned}\tag{C.25}$$

Although the linearised sSFCs of (C.25) are at least independent, they no longer constrain the required 2×10 canonical D.o.F. This immediate discord between the linear and nonlinear constraints is consistent with the results of Chapter 4, and as we show in Appendix C.8, it can be understood as a consequence of the missing Einstein–Hilbert term.

A careless multiplicity counting based on Eqs. (C.15a) to (C.15f) and Eqs. (C.24a) to (C.24d) then suggests that Case 16 nonlinearly propagates 13 D.o.F: a far cry from the two linear D.o.F predicted in [153]. Unless a more careful analysis leads to the discovery of FC PiC combinations which ultimately cull 11 D.o.F, we must conclude that the nonlinear unitarity of the theory is unsafe.

We also note that practical challenges arise even at the linear level. The velocities of the PiCs lead linearly to the following SiCs

$$\begin{aligned}
&\hat{\chi}^{\flat}{}_{\overline{ij}}\approx-N^{\flat}J^{\flat}\delta^{\flat\overline{k}}_{[\overline{i}}\mathcal{D}^{\flat}{}_{\overline{j}]}\hat{\pi}^{\flat}{}_{\perp\overline{k}},\quad \tilde{\chi}^{\flat}{}_{\overline{ij}}\approx-N^{\flat}J^{\flat}\delta^{\flat\overline{k}}_{\langle\overline{i}}\mathcal{D}^{\flat}{}_{\overline{j}\rangle}\hat{\pi}^{\flat}{}_{\perp\overline{k}},\\
&\chi^{\flat}{}_{\perp}\approx\frac{N^{\flat}}{J^{\flat}}\left(\hat{\pi}^{\flat}-\eta^{\flat\overline{ij}}\mathcal{D}^{\flat}{}_{\overline{i}}\vec{\hat{\pi}}^{\flat}{}_{\overline{j}}\right)+8N^{\flat}\hat{\alpha}_6\eta^{\flat\overline{ij}}\mathcal{D}^{\flat}{}_{\overline{i}}\mathcal{R}^{\flat}{}_{\perp\overline{j}},\\
&{}^{\mathrm{T}}\chi^{\flat}{}_{\overline{ijk}}\approx\frac{2N^{\flat}}{J^{\flat}}{}^{\mathrm{T}}\check{\mathcal{P}}^{\flat}{}_{\overline{ijk}}{}^{\overline{pqr}}\mathcal{D}^{\flat}{}_{\overline{p}}\left(\hat{\hat{\pi}}^{\flat}{}_{\perp\overline{qr}}+\tilde{\hat{\pi}}^{\flat}{}_{\perp\overline{qr}}-16\hat{\alpha}_6 J^{\flat}\left(\underline{\mathcal{R}}^{\flat}{}_{[\overline{qr}]}+\underline{\mathcal{R}}^{\flat}{}_{\langle\overline{qr}\rangle}\right)\right),
\end{aligned}\tag{C.26}$$

of which $\chi^{\flat}{}_{\perp}$ and ${}^{\mathrm{T}}\chi^{\flat}{}_{\overline{ijk}}$ contain gradients of field strengths. In order to pursue the linear Dirac–Bergmann algorithm to completion and confirm [153], we would therefore need to extend to the second-order Euler–Lagrange formalism.

C.8 Complete analysis of the simple spin 1^+ case

It is useful to analyse the Hamiltonian structure of more 'conventional' PGT$^{q,+}$s which include the Einstein–Hilbert term through the parameter $\hat{\alpha}_0$, and which are to be contrasted with the purely quadratic theories considered in Chapters 2 to 5. As may be expected, the Einstein–Hilbert term essentially affects the constraint structure. The canonical definition of the PiC function (4.10a) associated with the 0^+ part of the rotational gauge field, and the super-Hamiltonian defined in (4.15a) both acquire new terms $\varphi_{\perp}\mapsto\varphi_{\perp}+3\hat{\alpha}_0 m_{\mathrm{p}}{}^2$ and $\mathcal{H}_{\perp}\mapsto\mathcal{H}_{\perp}+\frac{1}{2}\hat{\alpha}_0 m_{\mathrm{p}}{}^2 J\underline{\mathcal{R}}$, respectively of order unity and first perturbative order. We will consider in this appendix the 'minimal' modification of the ECT analysed previously in [169], which propagates a 1^+ tordion in addition to the graviton

$$\hat{\alpha}_1=\hat{\alpha}_2=\hat{\alpha}_3=\hat{\alpha}_4=\hat{\alpha}_6=\hat{\beta}_1=\hat{\beta}_2=0. \tag{C.27}$$

We note that the defining conditions in (C.27) are consistent with curvature-free constraints: the PiCs depend only on the momenta. We extend Appendix C.3 by listing the nonlinear commutators of this theory

$$
\begin{aligned}
&\Big\{\varphi, \varphi_{\perp}\Big\} \approx -\frac{6\hat{\alpha}_0}{J} {m_{\mathrm{p}}}^2\delta^3, \quad \Big\{\varphi_{\perp\bar{i}}, \varphi_{\perp\bar{l}}\Big\} \approx \frac{2}{J^2}\overset{\wedge}{\hat{\pi}}_{\bar{i}\bar{l}}\delta^3, \quad \Big\{\varphi_{\perp\bar{i}}, \varphi_{\perp}\Big\} \approx -\frac{1}{J^2}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{i}}\delta^3, \\
&\Big\{\varphi_{\perp\bar{i}}, {}^{\mathrm{P}}\varphi\Big\} \approx -\frac{2}{J^2}\eta^{\overline{lp}}\eta^{\overline{mq}}\epsilon_{\overline{ilm}\perp}\overset{\wedge}{\hat{\pi}}_{\perp\overline{pq}}\delta^3, \quad \Big\{\varphi_{\perp\bar{i}}, \tilde{\varphi}_{\perp\overline{lm}}\Big\} \approx \frac{1}{2J^2}\eta_{\bar{i}\langle\bar{l}}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{m}\rangle}\delta^3, \\
&\Big\{\varphi_{\perp\bar{i}}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\Big\} \approx \frac{1}{2J^2}\Big(\eta_{\overline{in}}\overset{\wedge}{\hat{\pi}}_{\perp\overline{lm}} + \eta_{\bar{i}[\bar{m}}\overset{\wedge}{\hat{\pi}}_{\perp\bar{n}]\bar{l}} + \frac{3}{2}\eta_{\bar{n}[\bar{m}}\overset{\wedge}{\hat{\pi}}_{\perp\bar{l}]\bar{i}}\Big)\delta^3, \\
&\Big\{\tilde{\varphi}_{\overline{ij}}, \tilde{\varphi}_{\overline{lm}}\Big\} \approx -\frac{2}{J^2}\eta_{(\bar{i}|(\bar{l}}\overset{\wedge}{\hat{\pi}}_{\bar{m})|\bar{j})}\delta^3, \quad \Big\{\tilde{\varphi}_{\overline{ij}}, \tilde{\varphi}_{\perp\overline{lm}}\Big\} \approx \Big(\frac{\hat{\alpha}_0}{J}{m_{\mathrm{p}}}^2\eta_{\bar{i}\langle\bar{l}}\eta_{\bar{m}\rangle\bar{j}} - \frac{1}{J^2}\eta_{(\bar{i}|(\bar{l}}\overset{\wedge}{\hat{\pi}}_{\perp\bar{m})|\bar{j})}\Big)\delta^3, \\
&\Big\{\tilde{\varphi}_{\overline{ij}}, {}^{\mathrm{T}}\varphi_{\overline{lmn}}\Big\} \approx -\frac{1}{J^2}\,{}^{\mathrm{T}}\check{\mathcal{P}}_{\overline{lmn}}{}^{\overline{pqr}}\eta_{\bar{r}\langle\bar{i}}\eta_{\bar{j}\rangle\bar{p}}\overset{\rightharpoonup}{\hat{\pi}}_{\bar{q}}\delta^3.
\end{aligned}
\tag{C.28}
$$

On the PiC shell, the linearised sSFCs of the minimal theory are

$$
\begin{aligned}
&\mathcal{H}^{\flat}{}_{\perp} \approx \frac{1}{2}\hat{\alpha}_0{m_{\mathrm{p}}}^2J\underline{\mathcal{R}}^{\flat} \approx 0, \quad \mathcal{H}^{\flat}{}_{\bar{i}} \approx -\eta^{\flat\overline{jk}}\mathcal{D}^{\flat}{}_{\bar{j}}\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\overline{ik}} - \hat{\alpha}_0{m_{\mathrm{p}}}^2J\mathcal{R}^{\flat}{}_{\perp\bar{i}} \approx 0, \\
&\mathcal{H}^{\flat}{}_{\overline{ij}} \approx \mathcal{D}^{\flat}{}_{[\bar{i}}\overset{\rightharpoonup}{\hat{\pi}}{}^{\flat}{}_{\bar{j}]} + 2\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\overline{ij}} - \hat{\alpha}_0{m_{\mathrm{p}}}^2J\mathcal{T}^{\flat}{}_{\perp\overline{ij}} \approx 0, \quad \mathcal{H}^{\flat}{}_{\perp\bar{i}} \approx \eta^{\flat\overline{jk}}\mathcal{D}^{\flat}{}_{\bar{j}}\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\perp\overline{ik}} - \hat{\alpha}_0{m_{\mathrm{p}}}^2J^{\flat}\overset{\rightharpoonup}{\mathcal{T}}{}^{\flat}{}_{\bar{i}} \approx 0.
\end{aligned}
\tag{C.29}
$$

We observe in (C.29) a substantive departure from the sSFCs of the quadratic theories in Chapters 4 and 5. The Einstein–Hilbert term contributes independent parts of the Riemann–Cartan and torsion tensors to each irrep equation, thus subtracting 2×10 canonical D.o.F even at the linear level.

Some of the commutators in (C.28) also survive at the linear level, so that we do not have to worry about the consistency conditions of $\varphi^{\flat}$, $\tilde{\varphi}^{\flat}{}_{\overline{kl}}$, ${\varphi_{\perp}}^{\flat}$ and $\tilde{\varphi}^{\flat}{}_{\perp\overline{kl}}$. The consistencies of $\varphi^{\flat}{}_{\perp\bar{k}}$ and ${}^{\mathrm{P}}\varphi^{\flat}$ suggest the following secondaries on the combined shell of PiCs and sSFCs

$$
\chi^{\flat}{}_{\perp\bar{i}} \approx -\frac{2}{J^{\flat}}\eta^{\flat jk}\mathcal{D}^{\flat}{}_{\bar{j}}\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\overline{ik}} - \frac{\hat{\alpha}_0{m_{\mathrm{p}}}^2}{5\hat{\alpha}_5J^{\flat}}\overset{\rightharpoonup}{\hat{\pi}}{}^{\flat}{}_{\bar{i}} + \hat{\alpha}_0{m_{\mathrm{p}}}^2\mathcal{R}^{\flat}{}_{\perp\bar{i}} \approx 0, \quad \Big\{\chi^{\flat}{}_{\perp\bar{i}}, \varphi^{\flat}{}_{\perp\bar{l}}\Big\} \approx -\frac{{\hat{\alpha}_0}{}^2{m_{\mathrm{p}}}^2}{2\hat{\alpha}_5J^{\flat}}\eta^{\flat}{}_{\overline{il}}\delta^3, \tag{C.30a}
$$

$$
{}^{\mathrm{P}}\chi^{\flat} \approx -\frac{2}{J^{\flat}}\epsilon^{\flat\overline{ijk}\perp}\mathcal{D}^{\flat}{}_{\bar{i}}\overset{\wedge}{\hat{\pi}}{}^{\flat}{}_{\perp\overline{jk}} - (\hat{\alpha}_0 - 8\hat{\beta}_3){m_{\mathrm{p}}}^2\,{}^{\mathrm{P}}\mathcal{T}^{\flat} \approx 0, \quad \Big\{{}^{\mathrm{P}}\chi^{\flat}, {}^{\mathrm{P}}\varphi^{\flat}\Big\} \approx -\frac{24{m_{\mathrm{p}}}^2(\hat{\alpha}_0 - 8\hat{\beta}_3)}{J^{\flat}}\delta^3, \tag{C.30b}
$$

where these SiCs are SC at $\mathcal{O}(1)$, since they fail to commute with the PiCs which invoke them. This is to be expected from the theory of conjugate pairs [226]. We will not obtain the secondary ${}^{\mathrm{T}}\chi^{\flat}{}_{\overline{klm}}$ deriving from ${}^{\mathrm{T}}\varphi^{\flat}{}_{\overline{klm}}$, since the square of the tensor part projection entails calculations which are difficult on paper, however we note that these quantities should also form a conjugate SC pair.

In the final counting therefore, all the PiCs and SiCs are SC, and the linear theory propagates a total of *five* D.o.F

$$
\begin{aligned}
5 = \frac{1}{2}\big(&80 - 2\times 10[\mathrm{sPFC}] - 2\times 10[\mathrm{sSFC}] - (1+3+5+1+1+5+5)[\mathrm{iPSC}] \\
&- (3+1+5)[\mathrm{iSSC}]\big).
\end{aligned}
\tag{C.31}
$$

These D.o.F are interpreted as the massless graviton and a massive vector mode, so that the findings of [169] are confirmed.

C.9 The surficial commutator

An ostensibly limiting factor in previous Hamiltonian analyses of the PGT [226, 169, 168] is the dependence of various commutators on the *spatial gradient* of the equal-time Dirac function. The coefficients of such gradients are generally gauge dependent, while standard texts [219, 67] do not (to our knowledge) provide a prescription for their covariant interpretation. In this appendix we provide the covariant extension to the 'Poisson bracket formula' (4.12), which eliminates these gradients through the use of surface terms. The resulting expression is more costly to evaluate than the original by a factor of only several, allowing us to proceed farther into the theory.

Our starting point is the realisation that the Poisson bracket is ultimately motivated by the time derivative operator. We consider the time derivative of the covariant quantity $\mathcal{A}_{\acute{u}}$, which is assumed to depend canonically on a collection of (matter or gravitational) fields $\{\phi^{\acute{w}}\}$ and their conjugate momenta $\{\pi_{\acute{w}}\}$, along with their *first* covariant derivatives $\{D_\mu\phi^{\acute{w}}\}$ and $\{D_\mu\pi_{\acute{w}}\}$. The total Hamiltonian is assumed to contain a term bilinear in two further covariant quantities $\mathcal{H}_{\mathrm{T}} \supset \mathcal{B}_{\acute{v}}\mathcal{C}^{\acute{v}}$, and so a simple algebra reveals that the velocity $\dot{\mathcal{A}}_{\acute{u}}$ contains terms of the form

$$\begin{aligned}\dot{\mathcal{A}}_{\acute{u}}(x_1) \supset \int \mathrm{d}^3x_2\{\mathcal{A}_{\acute{u}}(x_1),\mathcal{B}_{\acute{v}}(x_2)\}\mathcal{C}^{\acute{v}}(x_2) \equiv \Bigg[\left(\frac{\bar{\delta}\mathcal{A}_{\acute{u}}}{\bar{\delta}\phi^{\acute{w}}}\cdot\frac{\bar{\delta}\mathcal{B}_{\acute{v}}}{\bar{\delta}\pi_{\acute{w}}}-\frac{\bar{\delta}\mathcal{A}_{\acute{u}}}{\bar{\delta}\pi_{\acute{w}}}\cdot\frac{\bar{\delta}\mathcal{B}_{\acute{v}}}{\bar{\delta}\phi^{\acute{w}}}\right)\mathcal{C}^{\acute{v}}\\ +D_\alpha\left[\left(\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\phi^{\acute{w}}}\cdot\frac{\bar{\delta}\mathcal{B}_{\acute{v}}}{\bar{\delta}\pi_{\acute{w}}}-\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\pi_{\acute{w}}}\cdot\frac{\bar{\delta}\mathcal{B}_{\acute{v}}}{\bar{\delta}\phi^{\acute{w}}}\right)\mathcal{C}^{\acute{v}}\right]+\left(\frac{\partial\mathcal{B}_{\acute{v}}}{\partial D_\alpha\phi^{\acute{w}}}\cdot\frac{\bar{\delta}\mathcal{A}_{\acute{u}}}{\bar{\delta}\pi_{\acute{w}}}-\frac{\partial\mathcal{B}_{\acute{v}}}{\partial D_\alpha\pi_{\acute{w}}}\cdot\frac{\bar{\delta}\mathcal{A}_{\acute{u}}}{\bar{\delta}\phi^{\acute{w}}}\right)D_\alpha\mathcal{C}^{\acute{v}}\\ -D_\alpha\left[\left(\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\phi^{\acute{w}}}\cdot\frac{\partial\mathcal{B}_{\acute{v}}}{\partial D_\beta\pi_{\acute{w}}}-\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\pi_{\acute{w}}}\cdot\frac{\partial\mathcal{B}_{\acute{v}}}{\partial D_\beta\phi^{\acute{w}}}\right)D_\beta\mathcal{C}^{\acute{v}}\right]\Bigg]\Bigg|_{x_1},\end{aligned} \tag{C.32}$$

where the dot product sums over field species and we construct a derivative which naturally extends the variational derivative on a scalar Lagrangian to tensors of arbitrary rank

$$\frac{\bar{\delta}\mathcal{A}_{\acute{u}}}{\bar{\delta}\phi^{\acute{w}}} \equiv \frac{\bar{\partial}\mathcal{A}_{\acute{u}}}{\bar{\partial}\phi^{\acute{w}}} - D_\alpha\left(\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\phi^{\acute{w}}}\right). \tag{C.33}$$

In (C.33) the notation $\bar{\partial}/\bar{\partial}\phi^{\acute{w}}$ indicates that $D_\alpha\phi^{\acute{w}}$ is held constant when evaluating the partial derivative. It is only expressions such as (C.32) which must be covariant, and the operations $\bar{\partial}/\bar{\partial}\phi^{\acute{w}}$, $\partial/\partial D_\alpha\phi^{\acute{w}}$, $\bar{\delta}/\bar{\delta}\phi^{\acute{w}}$ and their momentum counterparts all support that property. Therefore, we find it most natural to express the Poisson bracket as the *kernel* which reproduces (C.32). In general, this kernel takes the form of the second-order covariant differential operator

$$\begin{aligned}\{\mathcal{A}_{\acute{u}}(x_1),\mathcal{B}_{\acute{v}}(x_2)\} \equiv \left[\frac{\bar{\partial}\mathcal{A}_{\acute{u}}}{\bar{\partial}\phi^{\acute{w}}}\cdot\frac{\bar{\delta}\mathcal{B}_{\acute{v}}}{\bar{\delta}\pi_{\acute{w}}}-\frac{\bar{\partial}\mathcal{A}_{\acute{u}}}{\bar{\partial}\pi_{\acute{w}}}\cdot\frac{\bar{\delta}\mathcal{B}_{\acute{v}}}{\bar{\delta}\phi^{\acute{w}}}+\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\phi^{\acute{w}}}\cdot D_\alpha\left(\frac{\bar{\delta}\mathcal{B}_{\acute{v}}}{\bar{\delta}\pi_{\acute{w}}}\right)-\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\pi_{\acute{w}}}\cdot D_\alpha\left(\frac{\bar{\delta}\mathcal{B}_{\acute{v}}}{\bar{\delta}\phi^{\acute{w}}}\right)\right]\delta^3\\ +\left[\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\pi_{\acute{w}}}\cdot\frac{\bar{\delta}\mathcal{B}_{\acute{v}}}{\bar{\delta}\phi^{\acute{w}}}-\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\phi^{\acute{w}}}\cdot\frac{\bar{\delta}\mathcal{B}_{\acute{v}}}{\bar{\delta}\pi_{\acute{w}}}+\frac{\bar{\partial}\mathcal{A}_{\acute{u}}}{\bar{\partial}\pi_{\acute{w}}}\cdot\frac{\partial\mathcal{B}_{\acute{v}}}{\partial D_\alpha\phi^{\acute{w}}}-\frac{\bar{\partial}\mathcal{A}_{\acute{u}}}{\bar{\partial}\phi^{\acute{w}}}\cdot\frac{\partial\mathcal{B}_{\acute{v}}}{\partial D_\alpha\pi_{\acute{w}}}\right.\\ \left.+\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\beta\pi_{\acute{w}}}\cdot D_\beta\left(\frac{\partial\mathcal{B}_{\acute{v}}}{\partial D_\alpha\phi^{\acute{w}}}\right)-\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\beta\phi^{\acute{w}}}\cdot D_\beta\left(\frac{\partial\mathcal{B}_{\acute{v}}}{\partial D_\alpha\pi_{\acute{w}}}\right)\right]\delta^3 D_\alpha\\ +\left(\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\phi^{\acute{w}}}\cdot\frac{\partial\mathcal{B}_{\acute{v}}}{\partial D_\beta\pi_{\acute{w}}}-\frac{\partial\mathcal{A}_{\acute{u}}}{\partial D_\alpha\pi_{\acute{w}}}\cdot\frac{\partial\mathcal{B}_{\acute{v}}}{\partial D_\beta\phi^{\acute{w}}}\right)\delta^3 D_\alpha D_\beta.\end{aligned} \tag{C.34}$$

This concludes our discussion of the surficial commutator for the first-order Euler–Lagrange formalism.